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SUBJECT : DIFFERENTIAL EQUATIONS

SECTION : A

DATE : 20-8-2020

QUESTION NO # 1

$$\frac{dy}{dt} = e^{y \cdot t} \sec(y)(1+t^2)$$

$y(0)=0$ so $x=0$ $y=0$

$$dy = e^y \cdot e^{t} \sec(y)(1+t^2)\,dt$$

$$\frac{1}{e^y \cdot \sec(y)}\,dy = (1+t^2)e^{-t}dt$$

As $\cos(y) = \frac{1}{\sec(y)}$

$$\int e^{-y}\cos y\,dy = \int (1+t^2)e^{-t}dt$$

using integration by Parts

$$e^{-y}\int \cos y\,dx - \int\left(\int \cos y \cdot \frac{d}{dy}e^{-y}\right) =$$

$$(1+t^2)\int e^{-t} - \int\left(\int e^{-t} \cdot \frac{d}{dt}(1+t^2)\right)$$

eq (1)

L.H.S

$$e^{-y}\int \cos y\,dx - \int\left(\cos y\,\frac{d}{dy}e^{-y}\right.$$

$e^{-y}\sin y - \int (\sin y \cdot e^{-y}(-1))$

$e^{-y}\sin y + \int (\sin y \cdot e^{-y})$

$e^{-y}\sin y + \int (e^{-y}\sin y)$

Again using integration by parts

$e^{-y}\sin y + e^{-y}(-\cos y) - \int (\int \sin y \frac{d}{dy} e^{-y})$

$e^{-y}\sin y + e^{-y}(-\cos y) - \int (-\cos y \frac{e^{-y}}{-1})$

$e^{-y}\sin y - e^{-y}\cos y - \int (\cos y \, e^{-y})$

Since $\int (\cos y \, e^{-y}) = LHS$

L [illegible] = [illegible] 2

Now taking R.H.S

$\int (1+t^2)\, e^{-t} dt$

$= (1+t^2)\int e^{-t} - \int\left(\int e^{-t} \cdot \frac{d}{dt}(1+t^2)\right)$

$-(1+t^2)\, e^{-t} - \int(-e^{-t}(2t))$

$-(1+t^2)e^{-t} + \int((2t)e^{-t})$

Again using integration by parts

$-(1+t^2)e^{-t} + \left(2t\int e^{-t} - \int\left(\int e^{-t} \frac{d}{dt}\, 2t\right)\right)$

$-(1+t^2 e^{-t} + (-2te^{-t} - \int(-e^{-t}2)$

$-(1+t^2)e^{-t} + (-2te^{-t} + \int(2e^{-t}))$

$-(1+t^2)e^{-t} + (-2te^{-t} - 2e^{-t}) + c$

$= (1+t^2)e^{-t} - 2te^{-t} - 2e^{-t} + c$

$= e^{-t} - e^{-t}t^2 - 2te^{-t} - 2e^{-t} + c$

$= -(t^2+2t+3)e^{-t} + c = R.H.S$

Now take L.H.S = R.H.S

$$\frac{e^{-y}(\sin y - \cos y)}{2} = -(t^2+2t+3)e^{-t} + c$$

We know that

$t = 0 \quad y = 0$

Put it above

$= \frac{1}{2}(0-1) = -3 + c$

$c = 5/2$

Put value of c

(6)

$$\frac{e^{-y}}{2}(\sin y - \cos y) = -(x^2 + 2t + 3)e^{-t} + 5/2$$

Ans

7

Question No # 2

$(\sqrt{x+y}+\sqrt{x-y})dx - (\sqrt{x+y}-\sqrt{x-y})dy = 0$

$$\frac{dy}{dx} = \frac{\sqrt{x+y}+\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}} \longrightarrow (1)$$

This is homogenous differential eq in x and y to solve this $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Thus eq (1) becomes

$$v + x\frac{dv}{dx} = \frac{\sqrt{x+vx}+\sqrt{x-vx}}{\sqrt{x+vx}-\sqrt{x-vx}}$$

$$v + x\frac{dv}{dx} = \frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}$$

$$v + x\frac{dv}{dx} = \frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}} \times \frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}+\sqrt{1-v}}$$

$$v + x\frac{dv}{dx} = \frac{1+\cancel{v}+1-\cancel{v}+2\sqrt{1-v^2}}{2v}$$

$$v + x\frac{dv}{dx} = \frac{\cancel{2}(1+\sqrt{1-v^2})}{\cancel{2}v}$$

$$v + x\frac{dv}{dx} = \frac{1+\sqrt{1-v^2}}{v} - v$$

$$\cancel{x\frac{dv}{dx}} = \cancel{\frac{1+\sqrt{1-v^2}-v^2}{v}}$$

(8)

$$x\frac{dv}{dx} = \frac{1+\sqrt{1-v^2}-v^2}{v}$$

$$x\frac{dv}{dx} = \frac{\sqrt{1-v^2}\left(1+\sqrt{1-v^2}\right)}{v}$$

$$\frac{v\,dv}{\sqrt{1-v^2}\left(1+\sqrt{1-v^2}\right)} = \frac{dx}{x}$$

taking integral on b/s

$$\int \frac{v\,dv}{\sqrt{1-v^2}\left(1+\sqrt{1-v^2}\right)} = \int \frac{dx}{x}$$

Put $1+\sqrt{1-v^2} = t$

$$\Rightarrow \frac{1}{2}(1-v^2)^{-1/2}(-2v)\,dv = dt$$

$$\frac{v\,dv}{\sqrt{1-v^2}} = dt$$

$$\int -\frac{dt}{t} = \int \frac{dx}{x}$$

$$-\ln t = \ln x + \ln C$$

$$-\ln\left(1+\sqrt{1-v^2}\right) = \ln Cx$$

$$\ln\left(1+\sqrt{1-v^2}\right) = -\ln Cx$$

(9)

$$\cancel{\ln}\left(1+\sqrt{1-v^2}\right) = \cancel{\ln}(cx)^{-1}$$

$$1+\sqrt{1-v^2} = \frac{1}{cx}$$

$$1+\sqrt{1-\frac{y^2}{x^2}} = \frac{1}{cx}$$

$$1+\sqrt{\frac{x^2-y^2}{x^2}} = \frac{1}{cx}$$

$$x+\sqrt{x^2-y^2} = \frac{1}{c}$$

$$x+\sqrt{x^2-y^2} = c_1 \quad \because \frac{1}{c} = c_1$$

which is a required solution.

Question No # 3

$$(D^4+D^2)y = 3x^2 + 4\sin x - 2\cos x$$

Solution

$$(D^4+D^2)y = 3x^2 + 4\sin x - 2\cos x$$

$$\Rightarrow f(D)y = f(x)$$

As it is non-homogeneous linear equation So solution will be

$$y = y_c + y_p \quad -(i)$$

Complementary solution y_c

$$D^4+D^2 = 0 \Rightarrow D^2(D^2+1) = 0$$

Either $D^2 = 0 \Rightarrow \boxed{D = 0}$

$$D^2+1 = 0 \Rightarrow D^2 = -1$$

$$D = \sqrt{-1} \Rightarrow \boxed{D = i} \quad \text{or} \quad D = \boxed{0 \pm i}$$

Roots are equal and complex

$$y_c = c_1e^{0x} + c_2e^{0x}(c_2\cos x + c_3\sin x)$$

$$y_c = c_1 + c_2\cos x + c_3\sin x$$

$$y_p = \frac{1}{f(D)}F(x)$$

(1) Putting $D = 0$ in all

$$y_p = \frac{x^2 \cdot 3x^2}{12(0)+2} + \frac{x^2 \cdot 4\sin x}{12(0)+2} - \frac{2x \cdot 4\cos x}{12(0)+2}$$

$$y_p = \frac{3x^4}{2} + \frac{4x^2\sin x}{2} - \frac{2x^2\cos x}{2}$$

$$= \frac{3}{2}x^4 + 2x^2\sin x - x^2\cos x$$

So putting in equation (i)

$$y = c_1 + c_2\cos x + c_3\cos x + \frac{3}{2}x^4 + 2x^2\sin x$$
$$- x^2\cos x$$

$$y = c_1 + (c_2 - x^2)\cos x + (c_3 + 2x^2)\sin x + \frac{3}{2}x^4$$

(12)

$$y_P = \frac{1}{D^4+D^2}\left(3x^2+4\sin x-2\cos x\right)$$

$$= \frac{3x^2}{D^4+D^2} + \frac{4\sin x}{D^4+D^2} - \frac{2\cos x}{D^4+D^2}$$

$f(D) = D^4+D^2$

at $D=0 \Rightarrow f(D)=0$

so $f'(D) = 4D^3+2D$

Now also for $D=0 \Rightarrow f'(D)=0$

again diffrentiating

$f''(D) = 12D+2$

So for $D=0$

$f''(0) = 12(0)+2 = 2.$

So replacing $\frac{1}{f(D)}$ with $\frac{x^2}{f''(D)}$

$$\Rightarrow y_P = \frac{x^2\,3x^2}{12D+2} + \frac{x^2}{12D+2}\cdot 4\sin x - \frac{x^2}{12D+2}\cdot 2\cos x$$