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SUBJECT Differential Equation
DEPARTMENT BS (CS)
SEMESTER 4th

Q(1) (a)
Define differential equation along with two example?

A differential equation is an equation which contain one or more terms and the derivatives of one variable (i-e, dependent variable) with respect to the other variable (i-e, independent variable) $\frac{dy}{dx} = f(x)$ Here 'x' is an independent variable and 'y' is a dependent variable

For example:

(1) $$\frac{dy}{dx} = 5x$$

(2) $$\frac{dy}{dx} = 2x$$

Q(1) (b)

(b) Define a separable differential (SD) Equation?

A first order differential equation is said to be separable if, after solving it for the derivative, $\frac{dy}{dx} = F(x, y)$, the right-hand side can be factored as "a formula of just x" time "a formula of just y", $F(x,y) = f(x)g(y)$.

Q(1) part (b) (i)
Initial Value problem.

$$y' = \frac{xy^3}{\sqrt{1+x^2}} \quad y(0) = -1$$

First separate and then integrate both sides

$$y^{-3}dy = x(1+x^2)^{-\frac{1}{2}}dx$$

$$\int y^{-3}dy = \int x(1+x^2)^{-\frac{1}{2}}dx$$

$$-\frac{1}{2y^2} = \sqrt{1+x^2} + c$$

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Apply the intial condition to get the value of c

$$-\frac{1}{2} = \sqrt{1} + c \quad c = -\frac{3}{2}$$

implicit solution is

$$-\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

let solve for $y(x)$

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

$$y^2 = \frac{1}{3 - 2\sqrt{1+x^2}}$$

$$y(x) = \pm \frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

Reapplying the initial condition shows negative sign is the correct sign.

$$y(x) = -\frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}$$

since $1 + x^2 \geq 0$ the "inner" root will not be a problem. Therefore, all we need to

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$$= \quad 3 - 2\sqrt{1+x^2} > 0$$

$$= \quad 3 > 2\sqrt{1+x^2}$$

$$= \quad 9 > 4(1+x^2)$$

$$\frac{9}{4} > 1+x^2.$$

$$\frac{5}{4} > x^2.$$

<u>Ans.</u> $\boxed{-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}}$

Q(1) part (ii)
Solve the following for SD equations.

$$\frac{dx}{dt} = \frac{t}{x}.$$

Solution

$$\frac{dx}{dt} = \frac{t}{x}$$

$$\Rightarrow \quad x\,dx = t\,dt$$

Taking integration b/s.

$$\Rightarrow \int x\,dx = \int t\,dt$$

$$\Rightarrow \frac{x^2}{2} = \frac{t^2}{2} + C$$

$$\Rightarrow x^2 = t^2 + 2C$$

let $2C = C$

$$\boxed{x^2 = t^2 + C}$$ Ans.

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Q(2) part (a)

Explain the steps for solving linear Differential Equation.

Step by step method.

(1) Substitute $y = UV$
(2) Factor the parts involving V.
(3) Put the V term equal to zero.
(4) Solve using separation of variable to find U.
(5) Substitute U back into the equation we get at step 2.

(6) Solve that to find V.
(7) Finally, substitute U and V into $y = UV$ to get our solution.

Q(2) part (i)

$$\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1$$

$$y\left[\frac{\pi}{4}\right] = 3\sqrt{2} \qquad 0 \le x \le \frac{\pi}{2}$$

Solution:

$$y' + \frac{\sin(x)}{\cos(x)}y = 2\cos^2(x)\sin(x) - \frac{1}{\cos(x)}$$

$$y' + \tan(x)y = 2\cos^2(x)\sin(x) - \sec(x)$$

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Find the integrating factor

$$\mu(t) = e^{\int \tan(x)dx} = e^{\ln|\sec(x)|}$$

$$= e^{\ln \sec(x)} = \sec(x)$$

$$= \sec(x)$$

$$\int \tan(x)dx = -\ln|\cos(x)|$$

$$= \ln|\cos(x)|^{-1}$$

$$= \ln|\sec(x)|$$

$$e^{\ln f(x)} = f(x)$$

Multiply the integrating factor through the differential equation and verify the left side is a product rule.

$$\sec(x)y' + \sec(x)\tan(x)y = 2\sec(x)\cos^2(x)\sin(x) - \sec^2(x)$$

$$(\sec(x)y)' = 2\cos(x)\sin(x) - \sec^2(x)$$

integrate b/s.

$$\int (\sec(x)y(x))' dx = \int 2\cos(x)\sin(x) - \sec^2(x) dx$$

$$\sec(x)y(x) = \int \sin(2x) - \sec^2(x) dx$$

$$\sec(x)y(x) = -\frac{1}{2}\cos(2x) - \tan(x) + C$$

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Apply the initial condition to find the value of c.

$$3\sqrt{2} = y\left(\frac{\pi}{4}\right) = -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{4}\right) + c\cos\left(\frac{\pi}{4}\right)$$

$$3\sqrt{2} = -\frac{\sqrt{2}}{2} + c\,\frac{\sqrt{2}}{2}$$

$$c = 7$$

The solution is then

$$y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin x + 7\cos(x)$$

Q(2) part (v)

$$x' + 2x = \sin t.$$

Solution:-

$$\frac{dx}{dt} + 2x = 2\sin t \;B\;\_$$

Compare with $\frac{dy}{dx} + py = Q$

So.

$p = 2$ and $Q = 2\sin t$

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First we find Integration factor

~~$\int p\,dx$~~ ~~$\int 2\,dx$~~

$$\Rightarrow I.F = e^{\int p\,dx} = I.F\, e^{\int 2\,dx}$$

$$\Rightarrow \boxed{I.F = e^{2x}}$$

Now $x(I.F) = \int Q(I.F)\,dx + C$

putting values

$$\Rightarrow x.e^{2x} = \int 2\sin t . e^{2x} + C$$

$$\Rightarrow xe^{2x} = 2\int \sin t\, e^{2x} + C$$

$$\Rightarrow xe^{2x} = 2\left[e^{2x}\int \sin t\,dt - \int\left(\frac{d}{dx}e^{2x}\int \sin t\,dt\right)dt\right]$$

$$\Rightarrow xe^{2x} = 2\left[-e^{2x}\cos t - \int -2e^{2x}\cos t\,dt\right]$$

$$= xe^{2x} = 2\left[-e^{2x}\cos t + \left(e^{2x}\sin t\, t - 2xe^{2x}\right)\right]$$

$$\Rightarrow xe^{2x} = -2e^{2x}\cos t + 2e^{2x}\sin t - 4xe^{2x}$$

$$\Rightarrow 3x\cancel{e^{2x}} = 2\cancel{e^{2x}}(\sin t - \cos t)$$

$$\Rightarrow x = \frac{2}{3}(\sin t - \cos t)$$

Q (3) part (i)

$$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0 \ , \ y(0) = -3$$

Solution:

$$M = 2xy - 9x^2 \qquad M_y = 2x$$

$$N = 2y + x^2 + 1 \qquad N_x = 2x$$

actually we find $\psi(x,y)$.

$$\psi_x = M$$
$$\psi_y = N$$

$$\psi = \int M\,dx$$

OR

$$\psi = \int N\,dx$$

we will see the first one

$$\psi(x,y) = \int 2xy - 9x^2\,dx = x^2y - 3x^3 + h(y)$$

$$\psi_y = x^2 + h'(y) = 2y + x^2 + 1 = N$$

$$h'(y) = 2y + 1$$

now find $h(y)$ by integrating

$$h(y) = \int 2y + 1\,dy = y^2 + y + k$$

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$$\psi(x,y) = x^2y - 3x^3 + y^2 + y + k = y^2 + (x^2+1)y - 3x^3 + k$$

$$y^2 + (x^2+1)y - 3x^3 + k = c$$

$$y^2 + (x^2+1)y - 3x^3 = c - k$$

$$y^2 + (x^2+1)y - 3x^3 = c$$

apply the initial condition to find c.

$$(-3)^2 + (0+1)(-3) - 3(0)^3 = c \Rightarrow c = 6$$

$$y^2 + (x^2+1)y - 3x^3 - 6 = 0$$

we can solve for $y(x)$ by using the quadratic formula

$$y(x) = \frac{-(x^2+1) \pm \sqrt{(x^2+1)^2 - 4(1)(-3x^3-6)}}{2(1)}$$

$$= \frac{-(x^2+1) \pm \sqrt{x^4 + 12x^3 + 2x^2 + 25}}{2}$$

reapplying the initial condition

$$-3 = y(0) = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2}$$

$$= -3, 2$$

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$$y(x) = \frac{-(x^2+1) - \sqrt{x^4+12x^3+2x^2+25}}{2}$$

Now, for the interval of validity. It looks like we might well have problems with square roots of negative numbers. So we need to solve.

$$x^4 + 12x^3 + 2x^2 + 25 = 0$$

we get

$$x = -11.815 \text{ and}$$

$$x = -1.396$$

There are two interval where the polynomial will be positive

$$-\infty < x \leq -11.815$$

$$-1.396 \leq x < \infty$$

The interval of validity must be

$$-1.396 \leq x < \infty$$

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Q(3) part (ii)

$$\frac{2ty}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0 \quad y(5) = 0$$

Solution

$$\frac{2ty}{t^2+1} - 2t + (\ln(t^2+1) - 2)y' = 0$$

Find M and N

$$M = \frac{2ty}{t^2+1} - 2t \qquad M_y = \frac{2t}{t^2+1}$$

$$N = \ln(t^2+1) - 2 \qquad N_t = \frac{2t}{t^2+1}$$

Integrate the first one

$$\psi(t,y) = \int \frac{2ty}{t^2+1} - 2t\,dt = y\ln(t^2+1) - t^2 + h(y)$$

Differentiate with respect to y and compare to N

$$\psi_y = \ln(t^2+1) + h'(y) = \ln(t^2+1) - 2 = N$$

$$h'(y) = -2 \Rightarrow h(y) = -2y$$

$$\Psi(t,y) = y\ln(t^2+1) - t^2 - 2y$$

$$y\ln(t^2+1) - t^2 - 2y = c$$

Applying the initial condition gives

$$-25 = c$$

implicit solution is now.

$$y(\ln(t^2+1) - 2) - t^2 = -25$$

$$y(t) = \frac{t^2 - 25}{\ln(t^2+1) - 2}$$

$$\ln(t^2+1) - 2 = 0$$
$$\ln(t^2+1) = 2$$
$$t^2+1 = e^2$$

$$t = \pm\sqrt{e^2-1}$$

we have three possible intervals of validity

$$-\infty < t < -\sqrt{e^2-1}$$
$$-\sqrt{e^2-1} < t < \sqrt{e^2-1}$$

$$\sqrt{e^2-1} < t < \infty$$

The interval of validity of this problem is $\sqrt{e^2-1} < t < \infty$