Name: Muhammad Tayyab

ID # 7945

Section "B"

Fourth Semester

Subject: Differential Equation

Submitted To: Mam SHOMAILA MAZHER

DEPARTMENT OF Civil ENGINEERING

Q No#1: $x^3 y''' + 2x^2 y + 2y = 10x + \frac{10}{x}$

Solution:

$$\left(x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y\right) = 10x + \frac{10}{x}$$

$$\Rightarrow \left(x^3 \frac{d^3}{dx^3} + 2x^2 \frac{d^2}{dx^2} + 2\right) y = 10x + \frac{10}{x}$$

Put $D = \frac{d}{dx}$

$$\Rightarrow (x^3 D^3 + 2x^2 D^2 + 2)\, y = 10 + \frac{10}{x} \quad \text{---} (A)$$

Put $\rightarrow xD = \Delta$

$$= x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$$

$$= x^3 D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2\Delta$$

and $x = e^t$ in eq (A)

$$\Rightarrow (\Delta^3 - 3\Delta^2 + 2\Delta + 2\Delta^2 - \Delta + 2)\, y = 10e^t + \frac{10}{e^t}$$

$$\Rightarrow (\Delta^3 - \Delta^2 + 2)\, y = 10e^t + \frac{10}{e^t}$$

The characteristic equation is $\Delta^3 - \Delta^2 + 2 = 0$

Now by using synthetic division to find its Roots.

$$\begin{array}{r|rrrr} & 1 & -1 & 0 & 2 \\ -1 & & -1 & 2 & -2 \\ \hline & 1 & 2 & 2 & \boxed{0} \end{array}$$

$$\Rightarrow \Delta^2 - 2\Delta + 2 = 0$$

Now using Quadratic formula.

$$\Delta = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\Delta = \frac{-(-2) \pm \sqrt{b^2-4(1)(2)}}{2(1)}$$

$$\Delta = \frac{-(-2) \pm \sqrt{4-8}}{2}$$

$$\Rightarrow \frac{2 \pm \sqrt{-1} \times \sqrt{4}}{2}$$

$$\Rightarrow \frac{2 \pm \sqrt{2^2} \times i}{2}$$

$$\Rightarrow \frac{2 \pm 2i}{2}$$

$$\Rightarrow \frac{\cancel{2}(1 \pm i)}{\cancel{2}}$$

$\Delta = 1 \pm i$ ----> we take Just +ve on

Hence $m_1 = -1$, $m_2 = 1+i$

Since Root are Real and Complex.

(3)

$$y_c = c_1 e^{mt} + e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$$

Here $m =$ Real Number

$\alpha =$ Real part of complex Root

$B =$ Imaginary part of complet Root.

$$y_c = c_1 e^{-1} + e^{t}(c_2 \cos t + c_3 \sin t)$$

For $y_p = ?$

$$y_p = \frac{1}{\Delta^3 - \Delta^2 + 2} \cdot 10 e^{t} + \frac{1}{\Delta^3 - \Delta^2 + 2} \cdot \frac{10}{e^{t}}$$

$$\Rightarrow \frac{1}{(1)^3 - (1)^2 + 2} \cdot 10 e^{t} + \frac{1}{(-1)^3 - (-1)^2 + 2}$$ $\Rightarrow$ They becom zero so case fail

$\Rightarrow$ we take derarative

$$\Rightarrow \frac{1}{\cancel{1} - \cancel{1} + \cancel{2}} \cdot \cancel{10}^{5} e^{t} + \frac{(-)10 e^{-t}}{3\Delta^2 - 2\Delta}$$

$$\Rightarrow 5e^{t} - \frac{10 e^{-t}}{3(-1)^2 - 2(-1)}$$

$$\Rightarrow 5e^{t} - \frac{10 e^{-t}}{3+2} \Rightarrow 5e^{t} - \frac{\cancel{10}^{2} e^{-t}}{\cancel{5}_{1}}$$

$$y_p = 5e^{t} - 2e^{-t}$$

Now Genral Solution

(11)

Now General Solution

$$y = y_c + y_p$$

$$y = c_1 e^{-t} + e^{t}(c_2 \cos t + c_3 \sin t) + 5e^{t} - 2e^{-t}$$

Put $e^{t} = x$ and $t = \ln x$

$$y = c_1 x^{-1} + x(c_2 \cos \ln x + c_3 \sin \ln x) + 5x - 2x^{-1}$$

Q No # 02: $x^3 y''' + 4x^2 y'' - 5x y' - 15y = x^4$

Solution:. $\left(x^3 \frac{d^3}{dx^3} + 4x^2 \frac{d^2}{dx^2} - 5x \frac{d}{dx} - 15\right) y = x^4$

$\rightarrow$ put $D = \frac{d}{dx}$

$$\Rightarrow (x^3 D^3 + 4x^2 D^2 - 5xD - 15)\, y = x^4 \quad \text{---} \ (A)$$

$\rightarrow$ put $xD = \Delta$

$= x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

$= x^3 D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2\Delta$

and $x = e^{t}$ & $t = \ln x$ in eq (A)

$$\Rightarrow (\Delta^3 - 3\Delta^2 + 2\Delta + 4\Delta^2 - 4\Delta - 5\Delta - 15)\, y = e^{4t}$$

$$\Rightarrow (\Delta^3 + \Delta^2 - 7\Delta - 15)\, y = e^{4t}$$

The characteristic equation is $\Delta^3+\Delta^2-7\Delta-15=0$ using synthetic division to find its Roots.

$$\begin{array}{c|cccc} & 1 & +1 & -7 & -15 \\ 3 & & 3 & 12 & 15 \\ \hline & 1 & 4 & 5 & \boxed{0} \end{array}$$

$\Rightarrow \quad \Delta^2+4\Delta+5=0$

Now again to find at Roots we use Quadratic formula.

$$\Delta = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\Delta = \frac{-4 \pm \sqrt{4^2-4(1)(5)}}{2(1)}$$

$$\Delta = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{4}i}{2}$$

$$\Delta = \frac{-4+2i}{2} = \frac{2(-2 \pm i}{2}$$

$$\Delta = -2 \pm i$$

6

Hence $m_1 = 3$, $m_2 = -2 \pm i$

Since Root Are Real and complex

$$y_c = c_1 e^{mt} + e^{\alpha x}(c_2 \cos\beta t + c_3 \sin\beta t)$$

$$y_c = c_1 e^{3t} + e^{-2t}(c_2 \cos t + c_3 \sin t)$$

For $y_p = ?$

$$y_p = \frac{1}{D^3 + D^2 - 7D - 15} \cdot e^{4t}$$

$$y_p = \frac{1}{(4)^3 + (4)^2 - 7(4) - 15} \cdot e^{4t}$$

$$y_p = \frac{1}{80 - 43} \cdot e^{4t}$$

$$y_p = \frac{1}{37} \cdot e^{4t}$$

Hance The Genral Solution as

$$y = c_1 e^{3t} + e^{-2t}(c_2 \cos t + c_3 \sin t) + \frac{1}{37} \cdot e^{4t}$$

Replace $e^t = x$, $t = \ln x$

$$y = c_1 x^3 + x^{-2}(c_2 \cos \ln x + c_3 \sin \ln x) + \frac{1}{37} \cdot x^4$$

Q No # 03: $x^2 y'' + 2x y' - 6y = 10x^2$

Solution: $y(1) = 1$ and $y'(1) = -6$

$$\left(x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 6y = 10x^2\right.$$

$$\Rightarrow \left(x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} - 6\right) y = 10x^2$$

Put $xD = \Delta$,

$$x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$$

$$x = e^t, \quad t = \ln x$$

$$\Rightarrow (\Delta^2 - \Delta + 2\Delta - 6) y = 10 e^{2t}$$

$$\Rightarrow (\Delta^2 + \Delta - 6) y = 10 e^{2t}$$

The characteristic equation as.

$$\Delta^2 + \Delta - 6 = 0$$

By factorization

$$\Rightarrow \Delta^2 + 3\Delta - 2\Delta - 6 = 0$$

$$\Rightarrow \Delta(\Delta + 3) - 2(\Delta + 3) = 0$$

$$\Rightarrow (\Delta + 3)(\Delta - 2) = 0$$

$$\Rightarrow \Delta + 3 = 0, \quad \Delta - 2 = 0$$

$$\Delta = -3, \quad \Delta = 2$$

$$m_1 = -3 \quad m_2 = 2$$

Hance Root Are Real and distance.

For Perticuler Solution.

$$y_p = \frac{1}{\Delta^2+\Delta-6} \cdot 10e^{-2t}$$

$$y_p = \frac{10e^{2t}}{(2)^2+(2)-6} \Rightarrow \frac{10e^{2t}}{\cancel{6-6}}$$

Here case fail so we Apply L-Hospital Rule

$$y_p = \frac{(2)\,10e^{2t}}{2\Delta-1} \Rightarrow \frac{20e^{2t}}{2(2)-1}$$

$$y_p = \frac{20e^{2t}}{4-1} = \frac{20e^{2t}}{3}$$

$$y = y_c + y_p$$

$$y = c_1e^{-3t} + c_2e^{2t} + \frac{20}{3}e^{2t}$$

Replace $e^t = x$ and $t = \ln x$

$$\boxed{y = c_1x^{-3} + c_2x^2 + \frac{20}{3}x^2}$$

Now we find the value of c_1 and c_2 by initial conditions.

$y(1) = 1, \quad y(1) = -6$

$$y = c_1 x^{-3} + c_2 x^2 + \frac{20}{3}(x)^2 \quad \text{(A)}$$

$$y(1) = 1 \quad , \quad x = 1, \quad y = 1$$

$$1 = c_1 (1)^{-3} + c_2 (1)^2 + \frac{20}{3}(1)^2$$

$$1 = c_1 + c_2 + \frac{20}{3} \quad \text{(1)}$$

Now differentiate eq (A) w.r.t x

$$y' = -3 c_1 x^{-4} + 2 c_2 x + 2\left(\frac{20}{3}\right) x$$

$$y' = -3 c_1 x^{-4} + 2 c_2 x + \frac{40}{3} x$$

$$y'(1) = -6, \quad x = 1, \quad y = -6$$

$$-6 = -3 \cdot c_1 (1)^{-4} + 2 c_2 (1) + \frac{40}{3}(1)$$

$$-6 = -3 c_1 + 2 c_2 + \frac{40}{3} \quad \text{(2)}$$

Now eq (1) multiply by 2

$$2 = 2c_1 + 2c_2 + \frac{2(20)}{3}$$

$$\Rightarrow 2 = 2c_1 + 2c_1 + \frac{40}{3}$$

Then Eq (1) multiply by (2) and

Subtract eq (2) from Eq (1). We get.

$$2 = 2c_1 + 2c_2 + \frac{40}{3}$$

$$\pm 6 = \mp 3c_1 \pm 2c_2 + \frac{40}{3}$$

$$8 = 5c_1$$

$\boxed{c_1 = \frac{5}{8}}$ → Put in eq we get c_2

$$\Rightarrow 2 = 2\left(\frac{8}{5}\right) + 2c_2 + \frac{40}{3}$$

$$\Rightarrow 2 - 2\left(8/5\right) - \frac{40}{3} = 2c_2$$

$\Rightarrow 2 - \frac{16}{5} - \frac{40}{3} = 2c_2$ → By L.C.M

$$\Rightarrow \frac{30-48-200}{15} = 2c_2$$

$$\Rightarrow -\frac{218}{15} = 2c_2 \Rightarrow \frac{14.5}{2} = c_2$$

$$\boxed{c_2 = -7.2}$$

$$y = \frac{5}{8}x^{-3} + (-7.2)x^2 + \frac{20}{3}(x^2)$$

$y = \frac{5}{8}x^{-3} - 7.2x^2 + \frac{20}{3}x^2$ Required solution.

Q No#04: $x^2 y'' + 7xy + 5y = x^5$

Solution: $y(0) = 2$ and $y'(1) = 2$

$$= x^2 \frac{d^2y}{dx^2} + 7x \frac{dy}{dx} + 5y = x^5$$

$$= \left(x^2 \frac{d^2}{dx^2} + 7x \frac{d}{dx} + 5\right) y = x^5$$

Put $D = \frac{d}{dx}$

$$\Rightarrow (x^2 D^2 + 7xD + 5) y = x^5$$

Now put $xD = \Delta$

$$x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta \;\&\; x = e^t$$

$$= (\Delta^2 - \Delta + 7\Delta + 5) y = e^{5t}$$

$$= (\Delta^2 + 6\Delta + 5) y = e^{5t}$$

The characteristic quadratic formula we can find its Roots.

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1$, $b = +6$, $c = 5$

$$\Rightarrow \Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \Delta = \frac{-6 \pm \sqrt{-6^2 - 4(1)(5)}}{2(1)}$$

$$= \Delta = \frac{-6 \pm \sqrt{36 - 20}}{2} = \frac{-6 \pm \sqrt{16}}{2}$$

$$= \Delta = \frac{-6 \pm 4}{2} = \frac{2(-3 \pm 2)}{2} = -3 \pm 2$$

$$\Rightarrow \Delta_1 = -5 , \Delta_2 = -1$$

Since Root Are Real And distaince The Complementry Solution as:

for $y_c = c_1 e^{-5t} + c_2 e^{-t}$

For $y_p = ?$

$$\Rightarrow y_p = \frac{1}{\Delta^2 + 6\Delta + 5} \cdot e^{5t}$$

$$\Rightarrow y_p = \frac{1}{(5)^2 + 6(5) + 5} \cdot e^{5t}$$

$$\Rightarrow y_p = \frac{1}{25 + 30 + 5} e^{5t}$$

$$= y_p = \frac{1}{60} \cdot e^{5t}$$

Now The Genral Solution as

$$y = y_c + y_p$$

$$y = c_1 e^{-5t} + c_2 e^{-t} + \frac{1}{60} e^{5t} \cdots \text{(A)}$$

From The initial Condition we can find the value of c_1 and c_2. the initial Condition as $y(0) = 2$; $x = 0$ and $y = 2$ in eq (A) if we put OR Replace $e^{t} = x$ Then from initial Condition $x = 0$ When we put in eq (A) Right side become zero. After that we cannot find c_1 and c_2.

That is way we put $t = \ln x$ in eq (A) we get.

$$y = c_1 e^{-5\ln x} + c_2 e^{-\ln x} + \frac{1}{60} e^{5\ln x}$$

Now we put $y(0) = 2$; $x = 0$ $y = 2$

$$2 = c_1 e^{-5\ln(0)} + c_2 e^{-\ln(0)} + \frac{1}{60} e^{5\ln(0)}$$

$$2 = c_1 e^{-5} + c_2 e^{-1} + \frac{1}{60} e^{5} \because \ln(0) = 1$$

$$2 = 0.00673\, c_1 + 0.367\, c_2 + \frac{148}{12} \cdots (1)$$

Now differentiate eq (A) and put the initial conditions

$$y' = -5c_1 e^{-5t} - c_2 e^{-t} + \frac{5e^{5t}}{60}$$

Replace $x = e^t$

$$y' = -5c_1 x^{-5} - c_2 x^{-1} + \frac{5x^5}{60}$$

$$y'(1) = 2; \; x = 1, \; y' = 2$$

$$2 = -5c_1(1)^{-5} - c_2(1)^{-1} + \frac{5(1)^5}{60}$$

$$2 = 5c_1 - c_2 + \frac{\cancel{5}}{\cancel{60}_{12}}$$

$$2 = -5c_1 - c_2 + \frac{1}{12} \quad \text{——} \; (2)$$

Now eq (2) multiply by 0.367 and Then add eq + eq (1) we get

$$0.734 = -1.835c_1 - 0.\cancel{367}^{\,9}c_2 + \frac{0.367}{12}$$

$$2 = 0.00673c_1 + 0.\cancel{367}^{\,9}c_2 + \frac{148}{12}$$

$$2.734 = -1.7c_1 + 12.36$$

$$\Rightarrow 2.734 - 12.36 = -1.767c_1$$

$$c_1 = \frac{2.734 - 12.36}{-1.767} = 5.44$$

$$\boxed{c_1 = 5.44}$$

Hance $\Rightarrow 2 = 0.00673(5.44) + 0.367c_2 + \frac{148}{12}$

So

$$\Rightarrow 2 - 0.0073(5.44) - \frac{148}{12} = 0.367c_2$$

$$\Rightarrow c_2 = \frac{2 - 0.00673(5.44) - \frac{148}{12}}{0.367}$$

$$\boxed{c_2 = -28.2}$$

Put The value of c_1 and c_2 in genral Solution of equation.

$$y = 5.44e^{-5t} - 28.2e^{t} + \frac{1}{60}e^{5t}$$

Whien is Required Genral Solution

(16)

Q NO # 05 → $(x+1)^2 y'' - 3(x+1) y' + 4y = x^2$

Solution:

$$\Rightarrow (x+1)^2 \frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2$$

$$\Rightarrow \left((x+1)^2 \frac{d^2}{dx^2} - 3(x+1)\frac{d}{dx} + 4\right) y = x^2$$

Put $(x+1)(D) = \Delta$

$(x+1)^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

& $x = e^t \Rightarrow t = \ln x$

So first we put $D = \frac{d}{dx}$

$$\Rightarrow \left((x+1)^2 D^2 - 3(x+1) D + 4\right) y = x^2$$

$$\Rightarrow [\Delta^2 - \Delta - 3\Delta + 4]\, y = e^{2t}$$

$$\Rightarrow (\Delta^2 - 4\Delta + 4)\, y = e^{2t}$$

The characteristic eq is $\Delta^2 - 4D + 4 = 0$

for y_c we find the root By factorization

$$\Delta^2 - 4\Delta + 4 = 0$$

$$\Rightarrow \Delta^2 - 2\Delta - 2\Delta + 4 = 0$$

$$\Rightarrow \Delta(\Delta - 2) - 2(\Delta - 2) = 0$$

$\Rightarrow \Delta - 2 = 0 , \Delta - 2 = 0$

So $\Delta = +2 , \Delta = +2$

hance $m_1 = 2 , m_2 = 2$

Since Roots Are Real and Repeat

$y_c = (c_1 + c_2 t) e$

$y_c = (c_1 + c_2 t) e^{2t}$

for perticuler Soution $y_P = ?$

$$y_P = \frac{1}{\Delta^2 - 4\Delta + 4} e^{2t}$$

$$y_P = \frac{1}{(2)^2 - 4(2) + 4} \cdot e^{2t} = \frac{1}{4 - 8 + 4} \cdot e^{2t}$$

So we take deravative

$$y_P = \frac{2e^{2t}}{2\Delta - 4}$$

again if we put 2 Then comes zero So we take derivative

$$y_P = \frac{2 \cdot 2 e^{2t}}{2} = \frac{2\cancel{4} e^{2t}}{2} = 2e^{2t}$$

$$y = y_c + y_p$$

$$y = (c_1 + c_2 t)\, e^{2t} + 2e^{2t}$$

Replace $t = \ln x$ and $e^t = x$

$$y = (c_1 + c_2 \ln x)\, x^2 + 2x^2$$

Which is the Required Genral Solution.