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Sec- "B"

Subject – Differential Equations.

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Q.No. 01

Solve the following objective type questions

1) The order of the matrix A is $m \times p$ & the order of matrix B is $p \times n$ then the order of matrix AB is?

The order of matrix $AB = m \times n$.

2) The number of non-zero rows in Echelon form?

Rank of Matrix.

3) If $B = \begin{bmatrix} 1 & 4 \\ 2 & a \end{bmatrix}$ is singular matrix, than $a = ?$

$|B| = 0$ $\quad \begin{bmatrix} 1 & 4 \\ 2 & a \end{bmatrix}$

$|B| = 1 \times a - 4 \times 2$

$\Rightarrow a - 8 = 0$

$\Rightarrow a = 8$

iv) If $A = \begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$ then $|A| = ?$

$$|A| = \begin{vmatrix} 2i & i \\ i & -i \end{vmatrix}$$

$$= (2i)(-i) - (i)(i)$$

$$= -2i^2 - i^2$$

As, $i^2 = -1$

$$= -2(-1) - (-1)$$

$$\underline{|A| = 3.}$$

v) The matrix $A = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$ is?

<u>Scalor</u> <u>Matrix</u>.

vi) Solution of $\frac{dy}{dx} + 2xy = y$?

$$\frac{dy}{dx} + 2xy = y$$

$$\frac{dy}{dx} = y - 2xy$$

$$\frac{dy}{dx} = y(1 - 2x)$$

$$\int \frac{1}{y}\,dy = \int 1(1-2x)\,dx$$

$$\ln y = x - \frac{2x^2}{2} + c$$

$$\underline{y = e^{x-x^2} + c}.$$

vii) The order & degree of diffrential equation $\left(\frac{dy}{dx}\right)^3 = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ is

Order = 1

Degree = 6

viii) The order & degree of

$$\frac{d^2y}{dx^2} - 4xy = \sin\left(\frac{d^2y}{dx^2}\right) \text{ is ?}$$

Order = 2

Degree = Undefine.

ix) The differential equation

$$2\frac{dy}{dx} + x^2y = 2x+3, \quad y(0) = 5 \text{ is ?}$$

$$2\frac{dy}{dx} + x^2y = 2x + 3$$

$$\int 2\,dy = \int (2x + 3 - x^2y)\,dx$$

$$2y = \frac{2x^2}{2} + 3x - y\frac{x^3}{3} + C$$

$$y = \frac{x^2}{2} + \frac{3x}{2} - \frac{x^3y}{6} + C$$

Put $x = 0$, $y = 5$

$$5 = 0 + 0 - 0 + C$$

$C = 5$ <u>Homogenous Equation.</u>

So $$y = \frac{x^2}{2} + \frac{3x}{2} - \frac{x^3y}{6} + 5$$

x) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ is ?

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\underset{\sim}{R} \begin{vmatrix} 1 & a & a^2 \\ 1-1 & b-a & b^2-a^2 \\ 1-1 & c-a & c^2-a^2 \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

Expand by $C_1$

$$= 1 \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} - 0 + 0$$

$$= \{(b-a)(c^2-a^2)\} - \{(b^2-a^2)(c-a)\}$$

$$= (b-a)(c-a)(c-\cancel{a}+b+\cancel{a})$$

$$= (b-a)(c-a)(c-b).$$

Q. No(02)

(i) Express the determinant

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

As the product of factors which are linear in $a, b, c$?

Solution.

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Expand by $R_1$

$$= a\begin{vmatrix} b^2 & c^2 \\ b^3 & c^3 \end{vmatrix} - b\begin{vmatrix} a^2 & c^2 \\ a^3 & c^3 \end{vmatrix} + c\begin{vmatrix} a^2 & b^2 \\ a^3 & b^3 \end{vmatrix}$$

$$= a(b^2c^3 - b^3c^2) - b(a^2c^3 - a^3c^2) + c(a^2b^3 - a^3b^2)$$

$$= ab^2c^3 - ab^3c^2 - a^2bc^3 + a^3bc + a^2b^3c - a^3b^2c$$

Taking $(abc)$ common

$$= abc\,(bc^2 - b^2c - ac^2 + a^2c + ab^2 - a^2b)$$

$$= abc[bc(c-b) - ac(c-a) + ab(b-a)$$

Ans.

(ii) Find the Eigen Value

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

<u>Solution:</u>

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

Characteristic eq $|A - \lambda I| = 0$

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

take determinant

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & -1 & 0 \\ -1 & 3-\lambda & -1 & -1 \\ -1 & -1 & 3-\lambda & -1 \\ 0 & -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

Expand by $R_1$

$$\Rightarrow 2-\lambda \begin{vmatrix} 3-\lambda & -1 & -1 \\ -1 & 3-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 & -1 \\ -1 & 3-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} - 1 \begin{vmatrix} -1 & 3-\lambda & -1 \\ -1 & -1 & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0 - (1)$$

Again

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ -1 & 3-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix}$$ Expand by $R_1$.

$$\Rightarrow 3-\lambda \begin{vmatrix} 3-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} -1 & 3-\lambda \\ -1 & -1 \end{vmatrix}$$

$$= (3-\lambda)\left[(3-\lambda)(2-\lambda) - (-1)(-1)\right] + 1\left[(-1)(2-\lambda) - (-1)(-1)\right]$$
$$-1\left[(-1)(-1) - (-1)(3-\lambda)\right]$$

$$= (3-\lambda)(6-3\lambda-2\lambda+\lambda^2-1) + 1(-2+\lambda-1) - 1(1+3-\lambda)$$

$$= (3-\lambda)(\lambda^2-5\lambda+5) + (-3+\lambda) - (4-\lambda)$$

$$= 3\lambda^2 - 15\lambda + 15 - \lambda^3 + 5\lambda^2 - 5\lambda - 3 + \lambda - 4 + \lambda$$

$$= -\lambda^3 + 8\lambda^2 - 18\lambda + 8 \quad \text{— a}$$

Now

$$\Rightarrow +1 \begin{vmatrix} -1 & -1 & -1 \\ -1 & 3-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} \quad \text{Expand by } C_1$$

$$\Rightarrow -1 \begin{vmatrix} 3-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & 2-\lambda \end{vmatrix} + 0$$

$$\Rightarrow -1(6-3\lambda-2\lambda+\lambda^2-1) + (-2+\lambda-1)$$

$$\Rightarrow -\lambda^2 + 5\lambda - 5 - 3 + \lambda$$

$$\Rightarrow -\lambda^2 + 6\lambda - 8 \quad \text{— b}$$

$$\Rightarrow -1 \begin{vmatrix} -1 & 3-\lambda & -1 \\ -1 & -1 & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} \quad \text{Expand by } C_1.$$

$$\Rightarrow -1\left[-1\begin{vmatrix} -1 & -1 \\ -1 & 2-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 3-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} + 0\right]$$

$$= -\left[-(-2+\lambda-1) + 1(6-3\lambda-2\lambda+\lambda^2-1)\right]$$

$$= -(3-\lambda+\lambda^2-5\lambda+5)$$

$$= -\lambda^2+6\lambda-8 \quad \text{—} \; (c)$$

Put eq (a), (b) & (c) in eq (1)

$$(2-\lambda)(-\lambda^3+8\lambda^2-18\lambda+8) - \lambda^2+6\lambda-8-\lambda^2+6\lambda-8=0$$

$$= -2\lambda^3+16\lambda^2-36\lambda+16+\lambda^4-8\lambda^3+18\lambda^2-8\lambda-\lambda^2+6\lambda-8$$
$$-\lambda^2+6\lambda-8=0$$

$$= \lambda^4-2\lambda^3-8\lambda^3+16\lambda^2+18\lambda^2-\lambda^2-\lambda^2-36\lambda-8\lambda+6\lambda+6\lambda$$
$$+\cancel{16}-\cancel{8}-\cancel{8}=0$$

$$= \lambda^4-10\lambda^3+32\lambda^2-32\lambda=0.$$

Now, solve the upper equation by synthetic equation

| | | | | |
|---|---|---|---|---|
| | 1 | -10 | 32 | -32 |
| 2 | | | -16 | 32 |
| | 1 | -8 | 16 | 0 |

We get,

$$(\lambda-2)(\lambda^3-8\lambda+16\lambda)=0$$

$$\Rightarrow \lambda(\lambda-2)(\lambda^2-8\lambda+16)=0$$

| | | |
|---|---|---|
| $\lambda=0$ | $\lambda-2=0$ | $\lambda^2-8\lambda+16=0$ |
| | $\lambda=2$ | $\lambda^2-4\lambda-4\lambda+16=0$ |
| | | $\lambda(\lambda-4)-4(\lambda-4)=0$ |
| | | $(\lambda-4)(\lambda-4)=0$ |
| | | $\lambda=4,\ \lambda=4$ |

So

$\lambda_1=0$

$\lambda_2=2$

$\lambda_3=4$

$\lambda_4=4$ Ans.

Q.No.3

The rate of change in the form of differential equation is given by:

$$(x^2+3y^2)\,dx - 2xy\,dy = 0$$

Find the general solution at $x=2$ and $y=6$.

Solution:

$$(x^2+3y^2)\,dx - 2xy\,dy = 0$$

$$\Rightarrow (x^2+3y^2)\,dx = 2xy\,dy$$

$\div$ing both sides by $2xy\,dy$

$$\frac{dy}{dx} = \frac{x^2}{2xy} + \frac{3y^2}{2xy}$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{x}{y} + \frac{3y}{x}\right] \quad \text{— (A)}$$

Let $y = vx$

Diff:

$$dy = V dx + x\,dv$$

Dividing by $dx$

$$\frac{dy}{dx} = V + x\frac{dv}{dx} \quad \text{— (i)}$$

Put (i) in (A)

$$V + x\frac{dv}{dx} = \frac{1}{2}\left[\frac{\cancel{x}}{\cancel{x}V} + 3\frac{V\cancel{x}}{\cancel{x}}\right.$$

$$V + \frac{x\,dv}{dx} = \frac{1}{2}\left[\frac{1}{V} + 3V\right]$$

Xing both sides by "2"

$$2V + 2x\frac{dv}{dx} = \frac{1}{V} + 3V$$

$$2x\frac{dv}{dx} = \frac{1}{V} + 3V - 2V$$

$$2x\frac{dv}{dx} = \frac{1}{V} + V$$

$$2x\frac{dv}{dx} = \frac{1+V^2}{V}$$

Xing both sides by $\frac{dx}{dv}$

$$2x\,dv = \frac{1+v^2}{v}\,dx$$

Xing both sides by $\frac{v}{x(1+v^2)}$

$$\frac{v}{1+v^2}\,dv = \frac{1}{x}\,dx$$

Take "$\int$" on both sides

$$\int \frac{v}{1+v^2}\,dv = \int \frac{1}{x}\,dx + c$$

$$\ln|1+v^2| = \ln x + c$$

Take "e" both sides

$$e^{\ln|1+v^2|} = e^{\ln|xc|}$$

$$1+v^2 = xc$$

Put $v = y/x$

$$1 + (y/x)^2 = xc$$

$$\frac{x^2+y^2}{x^2} = xc$$

$$x^2+y^2 = x^3c \quad \text{— (B)}$$

Put $x=2, y=6$ in eq (B)

$$2^2+6^2 = 8c$$

$$40 = 8c$$

$$c = \cancel{\frac{40}{8}}\,5$$

$$c = 5$$

Put $c=5$ in eq (B)

$$x^2+y^2 = 5x^3$$

$$y^2 = 5x^3 - x^2$$

$$y^2 = x^2(5x-1)$$

Taking square root both sides

$$y = x\sqrt{5x-1}, \quad y = -x\sqrt{5x-1}$$

$$\boxed{y = \pm x\sqrt{5x-1}} \quad \underline{\text{Ans}}.$$