Name : Ahmed Musa

ID # 7944

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Submitted To : Mam SHOMAILA MAZHER

DEPARTMENT OF CIVIL ENGINEERING

IQRA NATIONAL UNIVERSITY PESHAWER

Q No #01 :→ $x^3y''' + 2x^2y + 2y = 10x + \frac{10}{x}$

Solution:→ $\left(x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y\right) = 10x + \frac{10}{x}$

$\Rightarrow \left(x^3\frac{d^3}{dx^3} + 2x^2\frac{d^2}{dx^2} + 2\right)y = 10x + \frac{10}{x}$

Put $D = d/dx$

$\Rightarrow (x^3D^3 + 2x^2D^2 + 2)y = 10 + \frac{10}{x}$ ——— (A)

Put → $xD = \Delta$

→ $x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

→ $x^3D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2\Delta$

and $x = e^t$ in eq (A)

$\Rightarrow (\Delta^3 - 3\Delta^2 + 2\Delta + 2\Delta^2 - 2\Delta + 2)y = 10e^t + \frac{10}{e^t}$

$\Rightarrow (\Delta^3 - \Delta^2 + 2)y = 10e^t + \frac{10}{e^t}$

The characteristic equation is $\Delta^3 - \Delta^2 + 2 = 0$

Now by using synthetic division to find its Roots.

$$\begin{array}{c|cccc} & 1 & -1 & 0 & 2 \\ -1 & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$\Rightarrow \Delta^2 - 2\Delta + 2 = 0$

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Now using Quadratic formula.

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Delta = \frac{-(-2) \pm \sqrt{b^2 - 4(1)(2)}}{2(1)}$$

$$\Delta = \frac{-(-2) \pm \sqrt{4 - 8}}{2}$$

$$\Rightarrow \frac{2 \pm \sqrt{-1} \times \sqrt{4}}{2}$$

$$\Rightarrow \frac{2 \pm \sqrt{2^2} \times i}{2}$$

$$\Rightarrow \frac{2 \pm 2i}{2}$$

$$\Rightarrow \frac{\cancel{2}(1 \pm i)}{\cancel{2}}$$

$\Delta = 1 \pm i \longrightarrow$ we take just +ve one

Hence $m_1 = -1$, $m_2 = 1 + i$

Since Root Are Real and Complex.

$$y_c = c_1 e^{mt} + e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$$

Here m = Real Number

α = Real part of Complex Root

β = Imaginary part of Complex Root.

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$$y_c = c_1 e^{-t} + e^{t}(c_2\cos t + c_3 \sin t)$$

For $y_p = ?$

$$y_p = \frac{1}{D^3 - D^2 + 2} \cdot 10e^{t} + \frac{1}{D^3 - D^2 + 2} \, 10e^{-t}$$

$$\Rightarrow \frac{1}{(1)^3 - (1)^2 + 2} \cdot 10e^{t} + \frac{1}{(-1)^3 - (-1)^2 + 2} \, 10e^{-t}$$

→ They beome zero so case fail → we take derivative.

$$\Rightarrow \frac{1}{\cancel{1}-\cancel{1}+\cancel{2}_1} \cdot \cancel{10}^{5} e^{t} + \frac{(-1)\, 10e^{-t}}{3D^2 - 2D}$$

$$\Rightarrow 5e^{t} - \frac{10e^{-t}}{3(-1)^2 - 2(-1)}$$

$$\Rightarrow 5e^{t} - \frac{10e^{-t}}{3+2} \Rightarrow 5e^{t} - \frac{\cancel{10}^{2} e^{-t}}{\cancel{5}_1}$$

$$y_p = 5e^{t} - 2e^{-t}$$

Now Genral Solution

$$y = y_c + y_p$$

$$y = c_1 e^{-t} + e^{t}(c_2 \cos t + c_3 \sin t) + 5e^{t} - 2e^{-t}$$

Put $e^{t} = x$ and $t = \ln x$

$$y = c_1 x^{-1} + x\left(c_2 \cos \ln x + c_3 \sin \ln x\right) + 5x - 2x^{-1}$$

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Q No # 02 $x^3y''' + 4x^2y'' - 5xy' - 15y = x^4$

Solution:→ $\left(x^3\frac{d}{dx^3} + 4x^2\frac{d_2}{dx^2} - 5x\frac{d}{dx} - 15\right)y = x^4$

→ Put $D = d/dx$

$\Rightarrow (x^3D^3 + 4x^2D^2 - 5xD - 15)y = x^4$ —— (A)

→ Put → $xD = \Delta$

→ $x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

→ $x^3D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2\Delta$

and $x = e^t$ & $t = \ln x$ in eq (A)

$\Rightarrow (\Delta^3 - 3\Delta^2 + 2\Delta + 4\Delta^2 - 4\Delta - 5\Delta - 15)y = e^{4t}$

$\Rightarrow (\Delta^3 + \Delta^2 - 7\Delta - 15)y = e^{4t}$

The characteristic equation is $\Delta^3 + \Delta^2 - 7\Delta - 15 = 0$

using synthetic division to find its Roots.

	1	+1	-7	-15
3		3	12	15
	1	4	5	0

$\Rightarrow \Delta^2 + 4\Delta + 5 = 0$

Now again to find at Roots

We use Quadratic formula.

⑤

$$\Delta = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\Delta = \frac{-4 \pm \sqrt{4^2-4(1)(5)}}{2(1)}$$

$$\Delta = \frac{-4 \pm \sqrt{16-20}}{2} \Rightarrow \frac{-4 \pm \sqrt{4}\, i}{2}$$

$$\Delta = \frac{-4 \pm 2i}{2} = \frac{\cancel{2}(-2 \pm i)}{\cancel{2}}$$

$$\Delta = -2 \pm i$$

Hance $m_1 = 3$, $m_2 = -2 \pm i$

Since Root Are Real and Complex

$$y_c = c_1 e^{mt} + e^{\alpha x}(c_2 \cos\beta t + c_3 \sin\beta t)$$

$$y_c = c_1 e^{3t} + e^{-2t}(c_2 \cos t + c_3 \sin t)$$

For $y_p = ?$

$$y_p = \frac{1}{\Delta^3+\Delta^2-7\Delta-15} \cdot e^{4t}$$

$$y_p = \frac{1}{(4)^3+(4)^2-7(4)-15} \cdot e^{4t}$$

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$$y_p = \frac{1}{64+16-28-15} \cdot e^{4t}$$

$$y_p = \frac{1}{80-43} \cdot e^{4t}$$

$$y_p = \frac{1}{37} \cdot e^{4t}$$

Hance The Genral solution as

$$y = y_c + y_p$$

$$y = c_1 e^{3t} + e^{-2t}(c_2 \cos t + c_3 \sin t) + \frac{1}{37} \cdot e^{4t}$$

** Replace $e^t = x$, $t = \ln x$

$$y = c_1 x^3 + x^{-2}(c_2 \cos \ln x + c_3 \sin \ln x) + \frac{1}{37} \cdot x^4$$

Q.No #03 $\rightarrow x^2 y'' + 2xy' - 6y = 10x^2$

$y(1) = 1$ and $y'(1) = -6$

Solution: $\rightarrow \left(x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 6y\right) = 10x^2$

$$\Rightarrow \left(x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} - 6\right) y = 10x^2$$

Put $xD = \Delta$,

$x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

$x = e^t, \quad t = \ln x$

$\Rightarrow (\Delta^2 - \Delta + 2\Delta - 6)y = 10e^{2t}$

$\Rightarrow (\Delta^2 + \Delta - 6)y = 10e^{2t}$

The characteristic equation as.

$\Delta^2 + \Delta - 6 = 0$

By factorization

$\Rightarrow \Delta^2 + 3\Delta - 2\Delta - 6 = 0$

$\Rightarrow \Delta(\Delta+3) - 2(\Delta+3) = 0$

$\Rightarrow (\Delta+3)(\Delta-2) = 0$

$\Rightarrow \Delta + 3 = 0, \quad \Delta - 2 = 0$

$\Delta = -3, \quad \Delta = 2$

$m_1 = -3, m_2 = 2$

Hance Root Are Real and distinct.

For $y_c = ?$

$y_c = C_1 e^{mt} + C_2 e^{mt}$

$y_c = C_1 e^{-3t} + C_2 e^{2t}$

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For Perticuler Solution

$$y_p = \frac{1}{\Delta^2 + \Delta - 6} \cdot 10 e^{2t}$$

$$y_p = \frac{10e^{2t}}{(2)^2 + (2) - 6} \Rightarrow \frac{10e^{2t}}{\cancel{6} - \cancel{6}}$$

Here case fail so we apply L-Hospital Rule

$$y_p = \frac{(2)10e^{2t}}{2\Delta - 1} \Rightarrow \frac{20e^{2t}}{2(2) - 1}$$

$$y_p = \frac{20e^{2t}}{4-1} = \frac{20e^{2t}}{3}$$

$$y = y_c + y_p$$

$$y = c_1 e^{-3t} + c_2 e^{2t} + \frac{20}{3} e^{2t}$$

Replace $e^t = x$ and $t = \ln x$

$$\boxed{y = c_1 x^{-3} + c_2 x^2 + \frac{20}{3} x^2}$$

Now we find the value of c_1 and c_2 by initial conditions.

$$y(1) = 1, \quad y'(1) = -6$$

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$$y = c_1 x^{-3} + c_2 x^2 + \frac{20}{3}(x^2) \quad \text{——} \ (A)$$

$y(1) = 1 \quad , \quad x = 1, y = 1$

$$1 = c_1 (1)^{-3} + c_2 (1)^2 + \frac{20}{3}(1)^2$$

$$1 = c_1 + c_2 + \frac{20}{3} \quad \text{——} \ (1)$$

Now differentiate eq (A) w.r.t x

$$y' = -3 c_1 x^{-4} + 2c_2 x + 2\left(\frac{20}{3}\right)x$$

$$y' = -3 c_1 x^{-4} + 2 c_2 x + \frac{40}{3} x$$

$y'(1) = -6 \quad , x = 1, \ y = -6$

$$-6 = -3 c_1 (1)^{-4} + 2 c_2 (1) + \frac{40}{3}(1)$$

$$-6 = -3 c_1 + 2 c_2 + \frac{40}{3} \quad \text{——} \ (2)$$

Now eq (1) Multiply By $\underline{\underline{2}}$

$$2 = 2c_1 + 2c_2 + \frac{2(20)}{3}$$

$$\Rightarrow 2 = 2c_1 + 2c_2 + \frac{40}{3}$$

Then Eq (1) Multiply by (2) and Subtract eq (2) from Eq (1). we get.

$$2 = 2c_1 + \cancel{2c_2} + \frac{40}{3}$$

$$\mp 6 = \mp 3c_1 \pm \cancel{2c_2} \pm \frac{40}{3}$$

$$8 = 5c_1$$

$$\boxed{c_1 = \frac{5}{8}} \rightarrow \text{put in eq we get } c_2$$

$$\Rightarrow 2 = 2\left(\frac{8}{5}\right) + 2c_2 + \frac{40}{3}$$

$$\Rightarrow 2 - 2(8/5) - \frac{40}{3} = 2c_2$$

$$\Rightarrow 2 - \frac{16}{5} - \frac{40}{3} = 2c_2 \rightarrow \text{By L.C.M}$$

$$\Rightarrow \frac{30 - 48 - 200}{15} = 2c_2$$

$$\Rightarrow \frac{-218}{15} = 2c_2 \Rightarrow \frac{14.5}{2} = c_2$$

$$\boxed{c_2 = -7.2}$$

$$y = \frac{5}{8}x^{-3} + (-7.2)x^2 + \frac{20}{3}(x^2)$$

$$y = \frac{5}{8}x^{-3} - 7.2x^2 + \frac{20}{3}x^2 \rightarrow \text{Required Solution.}$$

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Q No # 04 $x^2y'' + 7xy' + 5y = x^5$

Solution:→ $y(0) = 2$ and $y'(1) = 2$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 5y = x^5$$

$$\Rightarrow \left(x^2\frac{d^2}{dx^2} + 7x\frac{d}{dx} + 5\right)y = x^5$$

Put $D = d/dx$

$$\Rightarrow (x^2D^2 + 7xD + 5)y = x^5$$

Now put $xD = \Delta$

$x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$ & $x = e^t$

$$\Rightarrow (\Delta^2 - \Delta + 7\Delta + 5)y = e^{5t}$$

$$\Rightarrow (\Delta^2 + 6\Delta + 5)y = e^{5t}$$

The Characteristic equation as $\Delta^2 + 6\Delta + 5 = 0$

Now By using Quadratic formula we can find its Roots.

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1,\ b = +6,\ c = 5$

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$$\Rightarrow \Delta = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \Delta = \frac{-6 \pm \sqrt{-6^2-4(1)(5)}}{2(1)}$$

$$\Rightarrow \Delta = \frac{-6 \pm \sqrt{36-20}}{2} = \frac{-6 \pm \sqrt{16}}{2}$$

$$\Rightarrow \Delta = \frac{-6 \pm 4}{2} = \frac{\cancel{2}(-3 \pm 2)}{\cancel{2}} = -3 \pm 2$$

$$\Rightarrow \Delta_1 = -5, \ \Delta_2 = -1$$

Since Root Are Real And distinct
The Complementry Solution as:

For $y_c = c_1 e^{-5t} + c_2 e^{-t}$

For $y_p = ?$

$$\Rightarrow y_p = \frac{1}{\Delta^2+6\Delta+5} \cdot e^{5t}$$

$$\Rightarrow y_p = \frac{1}{(5)^2+6(5)+5} \cdot e^{5t}$$

$$\Rightarrow y_p = \frac{1}{25+30+5} e^{5t}$$

$$\Rightarrow y_p = \frac{1}{60} \cdot e^{5t}$$

Now The Genral solution as

$$y = y_c + y_p$$

$$y = C_1 e^{-5t} + C_2 e^{-t} + \frac{1}{60} e^{5t} \quad \text{— (A)}$$

From The initial Condition we can find The value of C_1 and C_2. The initial condition as $y(0) = 2$; $x=0$, and $y=2$

In eq (A) if we put OR Replace $e^t = x$ Then from initial condition $x=0$ when we Put in eq (A) Then Right side become zero. After that we cannot find the C_1 and C_2.

That's way we put $t = \ln x$ in eq (A) we get.

$$y = C_1 e^{-5\ln x} + C_2 e^{-\ln x} + \frac{1}{60} e^{5\ln x}$$

Now we put $y(0)=2$; $x=0, y=2$

$$2 = C_1 e^{-5\ln(0)} + C_2 e^{-\ln(0)} + \frac{1}{60} e^{5\ln(0)}$$

$$2 = C_1 e^{-5} + C_2 e^{-1} + \frac{1}{60} e^{5} \quad \therefore \ln(0) = 1$$

$$2 = 0.00673\, C_1 + 0.367 C_2 + \frac{148}{12} \quad \text{— (1)}$$

Now differentiate eq (A) and put the initial Conditions.

$$y' = -5c_1 e^{-5t} - c_2 e^{-t} + \frac{5e^{5t}}{60}$$

Replace $x = e^t$

$$y' = -5c_1 x^{-5} - c_2 x^{-1} + \frac{5x^5}{60}$$

$y'(1) = 2; \quad x = 1, y' = 2$

$$2 = -5c_1 (1)^{-5} - c_2 (1)^{-1} + \frac{5(1)^5}{60}$$

$$2 = -5c_1 - c_2 + \frac{\cancel{5}}{\cancel{60}_{12}}$$

$$2 = -5c_1 - c_2 + 1/12 \quad \text{———} \quad (2)$$

Now eq (2) multiply by 0.367 and Than add eq(1) + eq(2) we get

$$0.734 = -1.835c_1 - 0.\cancel{367}c_2 + \frac{0.367}{12}$$

$$2 = 0.00673c_1 + 0.\cancel{367}c_2 + \frac{148}{12}$$

$$2.734 = -1.767c_1 + 12.36$$

$$\Rightarrow 2.734 - 12.36 = -1.767c_1$$

$$c_1 = \frac{2.734 - 12.36}{-1.767} = 5.44$$

$$\boxed{c_1 = 5.44}$$

Hence $\Rightarrow 2 = 0.00673(5.44) + 0.367\,c_2 + \frac{148}{12}$

So $\Rightarrow 2 - 0.00673(5.44) - \frac{148}{12} = 0.367\,c_2$

$$\Rightarrow c_2 = \frac{2 - 0.00673(5.44) - \frac{148}{12}}{0.367}$$

$$\boxed{c_2 = -28.2}$$

Put the value of c_1 and c_2 in genral solution of equation.

$$y = 5.44\,e^{-5t} - 28.2\,e^{-t} + \frac{1}{60}\,e^{5t}$$

Which is Required Genral Solution

Q No#05 → $(x+1)^2y'' - 3(x+1)y' + 4y = x^2$

Solution :→

$$\Rightarrow (x+1)^2 \frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2$$

$$\Rightarrow \left((x+1)^2 \frac{d^2}{dx^2} - 3(x+1)\frac{d}{dx} + 4\right)y = x^2$$

Put $(x+1)(D) = \Delta$

$(x+1)^2 D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

& $x = e^t \Rightarrow t = \ln x$

So first we put $D = d/dx$

$$\Rightarrow ((x+1)^2 D^2 - 3(x+1)D + 4)y = x^2$$

$$\Rightarrow [\Delta^2 - \Delta - 3\Delta + 4]\, y = e^{2t}$$

$$\Rightarrow (\Delta^2 - 4\Delta + 4)y = e^{2t}$$

The characteristic eq is $\Delta^2 - 4\Delta + 4 = 0$

for y_c we find the root By factorization

$$\Delta^2 - 4\Delta + 4 = 0$$

$$\Rightarrow \Delta^2 - 2\Delta - 2\Delta + 4 = 0$$

$$\Rightarrow \Delta(\Delta - 2) - 2(\Delta - 2) = 0$$

$\Rightarrow s-2=0, \; s-2=0$

so $\quad s=+2, \; s=+2$

Hance $m_1=2, \; m_2=2$

Since Roots Are Real and Repert

$$y_c=(c_1+c_2t)e^{mt}$$

$$y_c=(c_1+c_2t)e^{2t}$$

For Perticuler Solution $y_p=?$

$$y_p=\frac{1}{s^2-4s+4}e^{2t}$$

$$y_p=\frac{1}{(2)^2-4(2)+4}\cdot e^{2t}=\frac{1}{\cancel{4}-\cancel{8}+\cancel{4}}e^{2t}$$

So we take derivative

$$y_p=\frac{2e^{2t}}{2s-4}$$

again if we put 2 Then comes zero so we take derivative

$$y_p=\frac{2\cdot 2e^{2t}}{2}=\frac{2\cancel{4}e^{2t}}{\cancel{2}}\Rightarrow 2e^{2t}$$

$$y = y_c + y_p$$

$$y = (c_1 + c_2 t)\, e^{2t} + 2e^{2t}$$

Replace $t = \ln x$ and $e^t = x$

$$y = (c_1 + c_2 \ln x)\, x^2 + 2x^2$$

which is The Required Genral Solution.