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Subject = Differential Equations

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Question No # 1 (1)

(i) $u = \sin(x+ct) + \cos(2x+2ct)$

Given $\dfrac{\partial^2 w}{\partial t^2} = \dfrac{c^2 \partial^2 w}{\partial x^2}$ —— (1)

Now:-

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$= \frac{\partial}{\partial t}(\sin(x+ct)) + \frac{\partial}{\partial t}(\cos(2x+2ct))$$

$$\frac{\partial w}{\partial t} = c\cos(x+ct) - 2c\sin(2x+2ct)$$

Prove $$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left[c\cos(x+ct) - 2c\sin(2x+2ct)\right]$$

$$\frac{\partial^2 w}{\partial t^2} = -c^2\sin(x+ct) - 4c^2\cos(2x+2ct)$$

Now

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x+ct) + \cos(2x+2ct)\right]$$

$$\frac{\partial w}{\partial x} = \cos(x+ct) - 2\sin(2x+2ct)$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left[\cos(x+ct) - 2\sin(2x+2ct)\right]$$

$$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$$

(2)

(1) $\Longrightarrow$

$$-c^2\sin(x+ct) - 4c^2\cos(2x+2ct) = c^2\left[-\sin(x+ct) - 4\cos(2x+2ct)\right]$$

$$-c^2\sin(\cancel{x+ct}) - 4c^2\cos(\cancel{2x+2ct}) = -c^2\sin(\cancel{x+ct}) - 4c^2\cancel{\cos}(2x+2ct)$$

$0 = 0$ (Satisfied).

(ii)

$w = \tan(2x+ct)$

Prove

$$\frac{\partial w}{\partial t} = c\sec^2(2x+ct)$$

& $$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left(c\sec^2(2x+ct)\right)$$

$$= c\,c^2 \cdot 2\sec^2(2x+ct)\tan(2x+ct)$$

Now:-

$$\frac{\partial w}{\partial x} = 2\sec^2(2x+ct)$$

$$\frac{\partial^2 w}{\partial x^2} = 4\sec^2(2x+ct)\tan(2x+ct)$$

(1) $\Longrightarrow$

$$4c^2\sec^2(2x+\cancel{ct})\tan(2x+ct) = 4c^2\sec^2(2x+\cancel{ct})\tan(2x+ct)$$

$0 = 0$ satisfied.

Question No #2 (3)

Given function is;

$$f(x) = \begin{cases} x; & -\pi < x \le 0 \\ 2x; & 0 \le x \le \pi \end{cases}$$

We have to find the fourier co-efficients, a_0, a_n & b_n

Now:-

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx + \frac{1}{\pi}\int_{0}^{-\pi} 2x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{2}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[0 - \frac{\pi^2}{2}\right] + \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right]$$

$$a_0 = \frac{-\pi}{2} + \pi = \pi/2 \rightarrow (1)$$

$$a_0 = \pi/2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(x\cos nx\,dx) + \frac{1}{\pi}\int_{0}^{\pi}(2x\cos nx)dx$$

$$\Rightarrow \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right)-\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$ (4)

$$+\frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right)-\left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}$$

$$a_n=\frac{1}{\pi}\left[\frac{\cos(0)}{n^2}-\frac{\cos n\pi}{n^2}\right]+\frac{2}{\pi}\left[\frac{\cos n\pi}{n^2}-\frac{\cos(0)}{n^2}\right]$$

$$=\frac{1}{\pi}\left(\frac{1-(-1)^n+2(-1)^n-2}{n^2}\right)=\frac{(-1)^n-1}{\pi n^2}$$

So

$$\boxed{a_n=\begin{cases}\frac{-2}{\pi n^2} & ; \text{ if } n \text{ is odd}\\ 0 & ; \text{ if } n \text{ is even}\end{cases}} \rightarrow (2)$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0}x\sin nx\,dx+\frac{2}{\pi}\int_{0}^{\pi}x\sin nx\,dx$$

$$=\frac{1}{\pi}\left(x\left(-\frac{\cos nx}{n}\right)-\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$+\frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-\left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$b_n=\frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right]+\frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right]=\frac{-3\cos n\pi}{n}$$

$$=\frac{3(-1)^{n+1}}{n}$$

So the required fourier Series is; (5)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi}{4} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2x-1)x}{(2n-1)^2} + 3 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$$

Question No # 3 6

$$y'' - 4y' - 13y = 8\sin 3x \quad y(0) = 1$$
$$y'(0) = 2$$

Sol:-

Associated Homogeneous eq of (1) is:

$$y'' - 4y' + 13y = 0 \longrightarrow (2)$$

change (2) into Auxiliary equation

put $y = m$ in eq (2)

$$m^2 - 4m + 13 = 0$$

use quadratic formula

$a = 1$, $b = -4$, $c = 13$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$$

$$m = \frac{4 \pm \sqrt{16 - 52}}{2}$$

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$$m = \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm \sqrt{36}\, i}{2}$$

$$= \frac{4 \pm 6i}{2}$$

$$= 2 + 3i$$

$m_1 = 2 + 3i$

$m_2 = 2 - 3i$

$$y_c = e^{2x}(C_1 \cos 3x + C_2 \sin 3x) \rightarrow (A)$$

Let

$$y_1 = A \cos 3x + B \sin 3x \longrightarrow (X)$$

Differentiate with respect to "x"

$$y_p' = -3A \sin 3x + 3B \cos 3x$$

$\Rightarrow$ Again differential w.r.t "x"

$$y_p'' = -9A \cos 3x - 9B \sin 3x$$

put in eq (1)

$$\Rightarrow (-9A\cos 3x - 9B\sin 3x) - 4(-3A\sin 3x + 3B\cos 3x) + 13(A\cos 3x + B\sin 3x) = 8\sin 3x$$

$$\Rightarrow -9A\cos 3x - 12B\cos 3x + 13A\cos 3x - 9B\sin 3x + 12A\sin 3x + 13B\sin 3x = 8\sin 3x$$

$$\Rightarrow (-9A - 12B + 13A)\cos 3x + (-9B + 12A + 13B)\sin 3x = 8\sin 3x$$

$$\Rightarrow (4A - 12B)\cos 3x + (4B + 12A)\sin 3x = 8\sin 3x$$

Camparing co-efficient

$\sin 3x \Rightarrow 4B - 12A = 8 \longrightarrow (a)$

$\cos 3x \Rightarrow 4A - 12B = 0 \qquad 4A = 12B$

$\Rightarrow A = 3B \longrightarrow (b)$

Put B in eq (a)

$4B + 12(3B) = 8$

$4B + 36B = 8$

$40B = 8$

$B = 1/5 \longrightarrow (c)$

(9)

put c in eq (b)

$\Rightarrow A = 3/5 \rightarrow$ (d)

put (c) & (d) in (x)

$$y_p = \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x \quad \text{—— (B)}$$

The G. Sol is

$$y = y_c + y_p$$

$$y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x) + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x \rightarrow \text{(c)}$$

Now we need to find the values of c_1 & c_2 for this;

put $x=0$ and $y=1$ in eq (c)

$$1 = e^{x(2)}\left(c_1 \cos 3(0) + c_2 \sin 3(0)\right) + \frac{3}{5}\cos 3(0) + \frac{1}{5}\sin 3(0)$$

$$1 = c_1(1) + c_2(0) + \frac{3}{5}(1) + \frac{1}{5}(0)$$

$$1 = c_1 + 3/5$$

$$c_1 = 1 - \frac{3}{5}$$

$$c_1 = 2/5 \quad \text{—— (xx)}$$

Differentiate (C) w.r.t "x"

$$y' = C_1(2e^{2x}\cos 3x - 3e^{2x}\sin 3x) + C_2(2e^{2x}\sin 3x + 3e^{2x}\cos 3x)$$

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$$-\frac{6}{5}\sin x + \frac{3}{5}\cos 3x \longrightarrow (D)$$

put $y' = 2$, $x = 0$ in eq (D)

$$y' = C_1(2e^{2x}\cos 3x - 3e^{2x}\sin 3x) + C_2(2e^{2x}\sin 3x + 3e^{2x}\cos 3x)$$
$$-\frac{6}{5}\sin 3x + \frac{3}{5}\cos 3x$$

put $y' = 2$, $x = 0$

$$2 = C_1(2e^{2(0)}\cos 3(0) - 3e^{2(0)}\sin 3(0) + C_2(3e^{2(0)}\sin 3(0) + 3e^{2(0)}\cos 3(0))$$
$$-\frac{6}{5}\sin 3(0) + \frac{3}{5}\cos 3(0)$$

$$2 = C_1(2) + C_2(3) - 0 + \frac{3}{5}$$

$$2 = 2C_1 + 3C_2 + \frac{3}{5}$$

put $C_1 = \frac{2}{5}$

$$2 = \frac{4}{5} + 3C_2 + \frac{3}{5}$$

$$2 = \frac{7}{5} + 3C_2$$

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$$3C_2 = 2 - \frac{7}{5}$$

$$3C_2 = \frac{3}{5}$$

$$C_2 = \frac{3}{15} \text{ --- } xxx$$

put xx and xxx in eq (c)

$$y = e^{2x}\left(\tfrac{2}{5}\cos 3x + \frac{3}{15}\sin 3x\right) + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x$$

$$y = \frac{2}{5}e^{2x}\cos 3x + \frac{3}{15}e^{2x}\sin 3x + \frac{3}{5}\cos 3x + \frac{1}{5}\sin 3x$$

$\Rightarrow$ Required General Solution.

Question No # 4 (12)

Sol:-

It is already in Symbolic form

$$(D^2 - DD')z = \cos x \cos 2y \quad \text{— (a)}$$

put AE $D^2 - DD' = 0$

As we know

$\frac{D}{D'} = m$ i.e $D = m$, $D' = 1$

$\Rightarrow m^2 - m = 0$

$m = 0, 1$

Therefore $C.f = f_1(y) + f_2(y+x)$ from eq (a)

$$P.I = \frac{1}{D^2 - DD'} \cos x \cos 2y$$

$$= \frac{1}{2} \frac{1}{D^2 - DD'} 2\cos x \cos 2y$$

As

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$C.f = f_1(y-x) + x f_2(y-x)$$

$$P.I = \frac{1}{D^2 + 2DD' + D'^2}\left[2(y-x) + \sin(x-y)\right]$$

$$= \frac{1}{(D+D')^2}\left[2(y-x) + \sin(x-y)\right]$$ 13

By General Method

$m = -1$, $y = x = c$

$$= \frac{1}{D+D'}\left[2c + \sin(-c)\right] dx$$

$$\Rightarrow \frac{1}{D+D'}\left[2cx - (\sin c)x\right]$$

Replacing c by $y-x$

Again put $y - x = c$

$$= \int (2xc - x \sin c)\, dx \Rightarrow cx^2 - \frac{x^2}{2} \sin c$$

$$\Rightarrow x^2(y-x) - \frac{x^2}{2}\sin(y-x) = x^2y - x^3 + \frac{x^2}{2}\sin(x-y)$$

Hence the required Solution is

$$z = C.F + P.I = f_1(y-x) + xf_2(y-x) + x^2y - x^3 + \frac{1}{2}x^2 \sin(x-y)$$