DIFFERENTIAL EQUATIONS

Final Assignment
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Q 1: (a)

2nd Order Linear Homogenous / non-homogenous differential equation:

"A differential equation of any order is homogenous, if once all the terms involving the unknown function are collected together on one side of the equation, the other side is identically zero."

i.e

$$y'' - 2y' + y = 0$$ is a 2nd order homogenous differential equation.

A differential equation of any order is non-homogenous, if once all the terms involving the unknown functions are collected together on one side, the other side is not identically zero.

i.e

$$y'' - 2y' + y = u$$

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Q1: B: i $4y'' - 6y' + 7y = 0$

Finding the Complementary Solution first

$$4y^2 - 6y + 7 = 0$$

The roots are

$$\lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(4)(7)}}{2(4)}$$

$$\lambda = \frac{6 \pm \sqrt{-76}}{8}$$

$$\lambda = \frac{6}{8} \pm \frac{2\sqrt{19}\,i}{8}$$

$$\lambda = \frac{3}{4} \pm \frac{\sqrt{19}\,i}{4}$$

$$\lambda_1 = \frac{3}{4} + \frac{\sqrt{19}i}{4}, \quad \lambda_2 = \frac{3}{4} - \frac{\sqrt{19}i}{4}$$

The perticular Solution is

$$a_1(u) = e^{\lambda_1 x} \cos^{\lambda_1(u)}$$

$$y = c_1 e^{\frac{3u}{4}} \cos \frac{i\sqrt{19}(u)}{4} + e^{\frac{3u}{4}} \sin \frac{i\sqrt{19}(u)}{4} c_2$$

Answer

Q1 : B : (ii)

$$y'' - 4y' - 12y = 3e^{5x}$$

First finding the Complementary Solution

$$y'' - 4y' - 12y = 0$$

The roots are

$$r^2 - 4r - 12 = (r-6)(r+2) = 0$$

$$\Rightarrow r_1 = -2, \quad r_2 = 6$$

The Complementary Solution is

$$y_c(x) = c_1 e^{-2x} + c_2 e^{6x}$$

Now lets find the perticular Solution

$$y_p(x) = Ae^{5x}$$

Plugging into the differential equation

$$25Ae^{5x} - 4(5Ae^{5x}) - 12(Ae^{5x}) = 3e^{5x}$$

$$25 - 20 - 12 (Ae^{5x}) = 3e^{5x}$$

$$-7Ae^{5x} = 3e^{5x}$$

$-7A = 3 \Rightarrow A = -\frac{3}{7}$

$y_p(x) = -\frac{3}{7}e^{5x}$

Ans

Q2: (i) $16y'' - 40y' + 25y = 0 \quad y(0) = 3, \; y'(0) = -\frac{9}{4}$

Sol

The characteristic equation here is

$$16r^2 - 40r + 25 = 0$$

The roots are

$$r_{1,2} = 5/4$$

The general solution is

$$y(t) = c_1 e^{\frac{5t}{4}} + c_2 e^{\frac{5t}{4}}$$

Plugging in the initial condition.

$$3 = y(0) = c_1$$

$$-9/4 = y'(0) = 5/4 c_1 + c_2$$

$$c_1 = 3, \quad c_2 = -6$$

The solution to the IVP is

$$y(t) = 3e^{\frac{5t}{4}} - 6te^{\frac{5t}{4}}$$

Answer

Q2: (ii) $y''+14y'+49=0$ $\quad y(-4)=-1,\ y'(-4)=5$

Sol

The characteristic equation here is

$$r^2+14r+49=0$$

The roots aree $r_{1,2}=-7$

The genral Solution is

$$y(t)=c_1e^{-7t}+c_2te^{-7t}$$

$$y'(t)=-7c_1e^{-7t}+c_2e^{-7t}-7c_2te^{-7t}$$

Plugging in the initial Conditions

$$-1=y(-4)=c_1e^{28}-4c_2e^{28}$$

$$5=y'(-4)=-7c_1e^{28}+c_2e^{28}+28c_2e^{28}=-7c_1e^{28}+29c_2e^{28}$$

$$c_1=-9e^{-28},\quad c_2=-2e^{-28}$$

The Solution to the IVP is

$$y(t)=9e^{-28}e^{-7t}-2te^{-28}e^{-7t}$$

$$y(t)=9e^{-7(t+4)}-2te^{-7(t+4)}$$

Answer

Q2 (iii) $y'' - 4y' + 9y = 0, \quad y(0) = 0, \; y'(0) = -8$

Sol

The characteristic equation here is

$$r^2 - 4r + 9 = 0$$

The roots are

$$r_{1,2} = 2 \pm \sqrt{5}i$$

The general solution is

$$y(t) = c_1 e^{2t} \cos(\sqrt{5}t) + c_2 e^{2t} \sin(\sqrt{5}t)$$

So $$0 = y(0) = c_1$$

hence the Solution is

$$y(t) = c_2 e^{2t} \sin(\sqrt{5}t)$$

$$y'(t) = 2c_2 e^{2t} \sin(\sqrt{5}t) + \sqrt{5}c_2 e^{2t} \cos(\sqrt{5}t)$$

Now Applying the second condition

$$-8 = y'(0) = \sqrt{5}c_2 \Rightarrow c_2 = \frac{8}{\sqrt{5}}$$

hence $$y(t) = \frac{-8}{\sqrt{5}} e^{2t} \sin(\sqrt{5}t)$$ Answer

Q2: (iv): $y'' - 8y' + 17y = 0 \quad y(0) = -4, \; y'(0) = -1$

Sol

The charactristic equation here is

$$r^2 - 8r + 17 = 0$$

The roots are $\quad r_{1,2} = 4 \pm i$

The genral solution is

$$y(t) = c_1 e^{4t} \cos(t) + c_2 e^{4t} \sin(t)$$
$$y'(t) = 4c_1 e^{4t} \cos(t) - c_1 e^{4t} \sin(t)$$
$$+ 4c_2 e^{4t} \sin(t) + c_2 e^{4t} \cos(t)$$

Applying initial conditions

$$-4 = y(0) = c_1$$
$$-1 = y'(0) = 4c_1 + c_2$$

$$c_1 = -4, \quad c_2 = 15$$

The solution to this IVP is

$$y(t) = -4e^{4t} \cos(t) + 15e^{4t} \sin(t)$$

Answer

Q3: Define Lapse Transform with examples.

Lapse Transform:

Lapse transform is is integral transform that converts a function of real variable (t) to the function of complex variable (s).

i.e:

The lapse transform $\mathcal{L}$, of a function $f(t)$ is for $t \geq 0$ is defined by the following integral ove $0 - \infty$.

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

Senral example is

$$F(s) = \int^{+\infty} f(t) \cdot e^{-s \cdot t} \cdot dt$$

Q3:

(1) $f(t) = 6e^{-5t} + e^{3t} + 5t^3$

Sol

$$F(s) = 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s}$$

$$= \frac{6}{s+5} + \frac{1}{s-3} \; \frac{30}{s^4} - \frac{9}{s}$$

Ans

(2) $g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$

Sol $$G(s) = 4\frac{s}{s^2+4^2} - 9\frac{4}{s^2+4^2} + 2\frac{s}{s^2+10^2}$$

$$= \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100}$$ Ans

(3) $h(t) = e^{3t} + \cos(6t) - e^{3t}\cos(6t)$

Sol $$H(s) = \frac{1}{s-3} + \frac{s}{s^2+6^2} - \frac{s-3}{(s-3)^2+6^2}$$

$$= \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}$$ Ans

Q4 : (i) $y'' - 10y' + 9y = 5t \quad y(0) = -1, \; y'(0) = 2$

Sol

Taking the transform of every term in DE.

$$\mathcal{L}\{y''\} - 10\,\mathcal{L}\{y'\} + 9\,\mathcal{L}\{y\} = \mathcal{L}\{5t\}$$

Using appropriate formulas

$$s^2 Y(s) - s y(0) - y'(0) - 10\left(sY(s) - y(0)\right) + 9Y(s) = \frac{5}{s^2}$$

Plugging the initial condition

$$(s^2 - 10s + 9)\,Y(s) + s - 12 = \frac{5}{s^2}$$

Solving for $Y(s)$

$$Y(s) = \frac{5}{s^2(s-9)(s-1)} + \frac{12-s}{(s-9)(s-1)}$$

Combining the two terms

$$Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}$$

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The partial fraction decomposition is

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

Setting numerator equal gives

$$5 + 12s^2 - s^3 = As(s-9)(s-1) + B(s-9)(s-1) + Cs^2(s-1) + Ds^2(s-9)$$

Solving for constants

$s = 0 \qquad 5 = 9B \implies B = \frac{5}{9}$

$s = 1 \qquad 16 = -8D \implies D = -2$

$s = 9 \qquad 248 = 648C \implies C = \frac{31}{81}$

$s = 2 \qquad 45 = -14A + \frac{4345}{81} \implies A = \frac{50}{81}$

Plugging in the constants.

$$Y(s) = \frac{\frac{50}{80}}{s} + \frac{\frac{5}{9}}{s^2} + \frac{\frac{31}{81}}{s-9} - \frac{2}{s-1}$$

Solution to the IVP is

$$y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{81}{81}e^{9t} - 2e^{t}$$

Q4: (ii) $y'' - 6y' + 15y = 2\sin(3t)$

$y(0) = -1$, $y'(0) = -4$

Sol

~~tatti~~ Taking the Lapse transform

$$s^2 Y(s) - sy(0) - y'(0) - 6(sY(s) - y(0)) + 15Y(s) = 2\frac{3}{s^2+9}$$

Plug in

$$(s^2 - 6s + 15)Y(s) + s - 2 = \frac{6}{s^2+9}$$

Solving fo $Y(s)$

$$Y(s) = \frac{-s^3 + 2s^2 - 9s + 24}{(s^2+9)(s^2-6s+15)}$$

Now lets get the partial friction decomposition

$$Y(s) = \frac{As + B}{s^2+9} + \frac{Cs + D}{s^2 - 6s + 15}$$

Setting numerators equal gives

$$-s^3+2s^2-9s+24 = (As+B)(s^2-6s+15)+(Cs+D)(s^2+9)$$
$$= (A+C)s^3+(-6A+B+D)s^2+(15A-6B+9C)s+15B+9D$$

Solving for the constants.

$$\left.\begin{aligned} s^3&: A+C=-1 \\ s^2&: -6A+B+D=2 \\ s^1&: 15A-6B+9C=-9 \\ s^0&: 15B+9D=24 \end{aligned}\right\} \Rightarrow \begin{aligned} A&=\frac{1}{10}, & B&=\frac{1}{10} \\ C&=-\frac{11}{10}, & D&=\frac{5}{2} \end{aligned}$$

Plugging into the decomposition

$$Y(s) = \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11s+25}{s^2-6s+15}\right)$$

$$= \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11(s-3+3)+25}{(s-3)^2+6}\right)$$

$$= \frac{1}{10}\left(\frac{1}{s^2+9} + \frac{1\frac{3}{3}}{s^2+9} - \frac{11(s-3)}{(s-3)^2+6} - \frac{8\frac{\sqrt{6}}{\sqrt{6}}}{(s-3)^2+6}\right)$$

The solution to the IVP is

$$y(t) = \frac{1}{10}\left(\cos(3t)+\frac{1}{3}\sin(3t)-11e^{3t}\cos(\sqrt{6}t) - \frac{8}{\sqrt{6}}e^{3t}\sin(\sqrt{6}t)\right)$$

Answer