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Section :- A

Subject :- Differential equation

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Assigment :- 02

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cauchy euler method:

Question # 1

The Cauchy Euler method or equation.

① $x^3 y''' + 2x^2 y' + 2y - 10x + \frac{10}{x}$

Solution:-

$\Rightarrow x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{dy}{dx} + 2y = 10x + 10x^{-1}$

$\Rightarrow x^3 D^3 y + 2x^2 D^2 + 2y = 10x + 10x^{-1}$

$\Rightarrow (x^3 D^3 + 2x^2 D + 2) y = 10x + 10x^{-1} \Rightarrow (1)$

let $x = e^t \Rightarrow t = \ln x$

$xD = D$

$x^2 D^2 = D(\Delta - 1) = \Delta^2 - \Delta$

$x^3 D^3 = \Delta(\Delta - 1)(\Delta - 2)$

Subtituting into eq (1)

$(\Delta - 3\Delta^2 + 2\Delta + 2(\Delta^2 - \Delta) + 2) y = 10^x + 1$

$(\Delta^3 - \Delta^2 + 2)\, y = 10x + 10x^{-1}$

$(m^3 - m^2 + 2)\, y = 10e^t + \frac{10}{e^t}$

using synthetic division

$$\begin{array}{r|rrrr} & 1 & -1 & 0 & 2 \\ -1 & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$$\Delta^2 - 2\Delta + 2 = 0$$

Now using quadratic formula:-

$a = 1$, $b = -2$, $c = 2$

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Delta = \frac{-(-2) \pm \sqrt{-2^2 - 4(1)(2)}}{2a}$$

$$\Delta = \frac{2 \pm \sqrt{4-8}}{2}$$

$$\Delta = \frac{2 \pm \sqrt{-4}}{2}$$

$$D = \frac{2 \pm \sqrt{-4}}{2}$$

$$D = \frac{2 \pm \sqrt{-1} \times \sqrt{4}}{2}$$

$$D = \frac{2 \pm 2i}{2}$$

$$D = \frac{\cancel{2}(1 \pm i)}{\cancel{2}}$$

$$D = 1 \pm i$$

Since roots are Complex

$$y_c = e^{-x}(c_1 \cos t + c_2 \sin t)$$

Now Particular integration

$$y_p = \frac{1}{D^3 - D^2 + 2} \cdot 10e^t + \frac{1}{D^3 - D^2 + 2} \cdot 10/e^t$$

$$= \frac{10e^t}{(1)^3 - (1)^2 + 2} + \frac{10e^{-t}}{(1)^3 - (1)^2 + 2}$$

$$\Rightarrow \frac{\cancel{10}^{5}e^{t}}{\cancel{2}} + \frac{\cancel{10}^{5}e^{-t}}{\cancel{2}}$$

$$\Rightarrow 5e^{t} + 5e^{-t}$$

$$y_p = 5e^{t} + 5e^{-t}$$

General Solution: $y = y_c + y_p$

$$y = e^{-x}(c_1 \cos t + c_2 \sin t) + 5e^{t} + 5e^{-t}$$

put $e^{t} = x$ and $t = \ln x$

$$y = e^{-x}(c_1 \ln x + c_2 \sin \ln x) + 5e^{x} + 5e^{-x} \text{ Ans.}$$

Question:- 2

$$x^3 \frac{d^3y}{dx} + 4x^2 \frac{d^2y}{dx} - 5x\frac{dy}{dx} - 15y = x^4$$

Solution:- Let $\frac{d}{dx} = D$

$$x^3 D^3 y + 4x^2 D^2 y - 5x Dy - 15y = x^4$$

(5)

$(x^2 D^3 + 4x^2 D^2 - 5xD - 15)\, y = x^4$

let $x = e^t \Rightarrow t = \ln x$

$xD = \Delta$

$x^2 D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

$x^3 D^3 = \Delta(\Delta - 1)(\Delta - 2) = \Delta^3 - 3\Delta^2 + 2$

Now substituting

$(x^3 D^3 + 4x^2 D^2 - 5xD - 15)\, y = x^4$

$(\Delta^3 - 3\Delta^2 + 2\Delta + 4(\Delta^2 - \Delta) - 5(\Delta - 15$

$y = e^{4t}$

$(\Delta^3 + \Delta^2 - 7\Delta - 15)\, y = e^{4t}$

Synthetic division

	1	+1	-7	-15
5		3	12	15
	1	4	5	0

$\Delta^2 + 4\Delta + 5 = 0$

Quadratic formula

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2}$$

$$= \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$D = \frac{\cancel{2}(-2 \pm i)}{\cancel{2}}$$

$$y_c = e^{5x}(C_1 \cos t + C_2 \sin t)$$

For $y_p = ?$

$$y_p = \frac{1}{D^3 + D^2 - 7D - 15} e^{4t}$$

$$= \frac{1}{(4)^3 + (4^2) - 7(4) - 15} e^{4t}$$

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$$= \frac{1}{64+16-28-15} e^{4t}$$

$$= \frac{1}{80-43} e^{4t}$$

$$y_P = \frac{1}{37} e^{4t}$$

Hence:- $y = y_c + y_P$

$$y = (c_1 \cos t + c_2 \sin t) + \frac{1}{37} e^{4t}$$

again $t = \ln x$ and $x = \ln x$

$$y = e^{3x}(c_1 \cos \ln x + (c_2 \sin \log x) + \frac{1}{37} e^{4x} \text{ Ans.}$$

X———X

Question 3:- $x^2 y'' + 2xy' - 6y = 10x^2$

Solution:- $y(1) = 1$ and $y'(1) = -6$

(8)

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 6y = 10x^2$$

$\Rightarrow \left(x^2 d^2/dx^2 + 2x\, d/dx - 6\right) y = 10x^2$

$\Rightarrow \left(x^2 d^2/dx^2 + 2x\, d/dx - 6\right) = 10x^2$

put $xD = \Delta \Rightarrow x^2D^2 = \Delta(\Delta$

$= \Delta^2 - \Delta$

$x = e^t$ and $\log x = t$

$(\Delta^2 - \Delta + 2\Delta - 6) y = 10e^{2t}$

$(\Delta^2 + \Delta - 6)\ y = 10e^{2t}$

The characteristic equation

$\Delta^2 + \Delta - 6 = 0$

$\Delta^2 + 3\Delta - 2\Delta - 6 = 0$

$= \Delta(\Delta+3) - 2(\Delta+3) = 0$

$= (\Delta+3)(\Delta-2) = 0$

$\Delta + 3 = 0, \quad \Delta - 2 = 0$

$\Delta = 2, \quad \Delta = -3$

Since roots are real and distinct for $y_c = ?$

$$y_c = c_1 e^{-3t} + c_2 e^{2t}$$

for $y_p = ?$

$$y_p = \frac{1}{\Delta^2 - \Delta - 6} \cdot 10^{2t}$$

$$= \frac{10}{\Delta^2 - \Delta - 6} e^{2t}$$

$$= 10 \frac{1}{0} e^{2t}$$

Now

$$10 \frac{1}{d/d\Delta(\Delta^2 + \Delta - 6)} e^{2t}$$

(10)

$\Rightarrow 10 \dfrac{t}{2\Delta+1} e^{2t}$

$= 10 \dfrac{1-t}{4+1} e^{2t}$

$y_p = 2t\, e^{2t}$

General solution

$y = y_c + y_p$

$= c_1 e^{-3t} + c_2 e^{2t} + 2t\, e^{2t}$

$y = c_1 x^{-3} + c_2 x^2 + 2(\log x) x^2$

put $y(1) = 1$ i.e $x = 1, y = 1$ i

$1 = c_1 (1)^{-3} + c_2 (1)^2 + 2 \log(1$

$1 = c_1 + c_2 \longrightarrow$ (c)

Now differentiate eq (B) w.r.t. x.

$y' = -3c_1 x^{-4} + 2c_2 x + \frac{2}{x}(x^2) + 4x\log x$

Now put $y'(1) = -6$ i.e $y' = -6$ and $x = -6$

$-6 = -3c_1 + 2c_2 + 2 + 0$

$\Rightarrow -6 = -3c_1 + 2c_2 + 2$

$\Rightarrow -6 - 2 = -3c_1 + 2c_2 + 2$

$-8 = -3c_1 + 2c_2 \rightarrow (1)$

×ing eq (1) with (2) & ÷ing

from (D)

$$\begin{aligned} 2c_1 + 2c_2 &= 2 \\ -3c_1 \pm 2c_2 &= \mp 8 \\ \hline 5c_1 &= 10 \end{aligned}$$

$c_1 = \cancel{10}^{2} / \cancel{5}_{1}$ $\boxed{c_1 = 2}$

$-8 = -3(2) + 2c_2$

$-8 = -6 + 2c_2$

$2c_2 = -8 + 6$

$2c_2 = -2$

$C_2 = \frac{-\cancel{2}}{2}\,1$

$\boxed{C_2 = -1}$

Now put the value of C_1 and C_2 in eq (B)

$y = 2x^{-3} - x^2 + 2\ln x \; x(x^2)$

$y = \frac{2}{x^3} - x^2 + 2x^2 \log x$ Ans

Question # 04

$x^2 y'' + 7xy' + 5y = x^5$

$y(0) = 2$ and $y'(1) = 2$

Solution:-

$x^2 \frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 5y = x^5$

$\Rightarrow \left(x^2 \frac{d^2}{dx^2} + 7x\frac{d}{dx} + 5\right)y = x^5$ - (A)

put $xD = \Delta \Rightarrow x^2D^2 = D(D-1)$
$= D^2 - D$

$x = e^t \Rightarrow \log x = t$ in eq (A)

$= (\Delta^2 - \Delta + 7\Delta + 5)\, y = e^{5t}$

By Chloraddatic formula

$$\Delta = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Delta = \frac{-6 \pm \sqrt{6^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{36 - 20}}{2}$$

$$= \frac{-6 \pm \sqrt{16}}{2}$$

$$= \frac{-6 \pm \sqrt{4^2}}{2}$$

$$= \frac{\cancel{2}(-3 \pm 2)}{\cancel{2}}$$

$\Delta = -3 \pm 2$ Since roots are real and distinct

$y_c = c_1 e^{-5t} + c_2 e^{-t}$

for $y_P = ?$

$$y_P = \frac{1}{D^2+6D+5} e^{5t}$$

$$= \frac{1}{(5)^2+6(5)+5} e^{5t}$$

$$= \frac{1}{60} e^{5t}$$

Now gendal Solution is

$$y = y_c + y_P$$

$$y = c_1 e^{-5t} + c_2 e^{-t} + \frac{1}{60} e^{5t}$$

$$y = c_1 x^{-5} + c_2 x^{-1} + \frac{1}{60} x^5 \rightarrow (B)$$

$x=0$ put in this equation

No in eq(B) $e^0 = 1$

put $y(0) = 2$ i-e $y = 2$ and $x = 2$

$$2 = c_1(2)^{-5} + c_2(2)^{-1} + \frac{1}{60}(2)^5$$

$\Rightarrow 2 = -32 c_1 - 2 c_2 + \frac{1}{6\cancel{0},5}(\cancel{32}^{8})$

$\Rightarrow 2 = -32 c_1 - 2 c_2 + \frac{8}{15}$

$\Rightarrow 2 - \frac{8}{15} = -32 c_1 - 2 c_2$

$\frac{22}{15} = -32 c_1 - 2 c_2 \rightarrow (c)$

Now differential equation (B) w.r.t (x)

$y' = -5c_1 x^{-6} - c_2 x^{-2} + \frac{1}{12} x^4$

$\rightarrow$ put $y'(1) = 2$ i-e $y' = 2$ and $x = 2$ in above equation

$2 = 5c_1 c_2 - c_1 - c_2 (2)^{-2} + \frac{1}{12}(2$

$2 = -5 c_1 (-64) - c_2 (4) + \frac{1}{12}(16)$

$2 = 320 c_1 + 4 c_2 + 4/3$

$$= 2 - \frac{4}{3} = 320 c_1 + 4 c_2$$

$$\Rightarrow 2/3 = 320 c_1 + 4 c_2 \rightarrow (D)$$

$\times$ing eq (C) with 2 and and then $-$ing eq (C) from (D)

$$\frac{-44}{15} = 64 + 4 c_2$$

$$\frac{-44}{15} = 64 c_1 + 4 c_1$$

$$\underline{+ 2/3 = +320 c_1 \pm 4 c_2}$$

$$\frac{34}{15} = -256 c_1$$

$$c_1 = \frac{34}{15} \times 256$$

$$\boxed{c_1 = 580}$$

put the value of c_1 in eq (C)

$$\frac{22}{15} = -32(580) - 2 c_2$$

$\Rightarrow \frac{22}{15} = -18560 - 2c_2$

$\Rightarrow \frac{22}{15} + 18560 = -2c_2$

$\Rightarrow \frac{18561}{2} = c_2$

$$\boxed{-9280 = c_2}$$

Now put the value of c_1 and c_2 in eq (B)

$y = 580x^{-5} - 9280x^{-1} + \frac{1}{60}x^5$

$y = \frac{580}{x^5} - \frac{9280}{x} + \frac{1}{60}x^5$

Ans.

Question 05: $(x+1)^2 y'' - 3(x+1)y' + 4y$

Solution:- $(x+1)^2 \frac{d^2y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2$

$= \left((x+1)^2 \frac{d^2}{dx^2} - 3(x+1)\frac{d}{dx} + 4\right) y = x^2$

$\Rightarrow \left[(x+1)^2 \frac{d^2}{dx^2} - 3(x+1)\frac{d}{dx} + 4\right) y = x^2$

$\Rightarrow \left[(x+1)^2 D^2 - 3(x+1) D + 4\right] y = x^2 \rightarrow$

put $(x+1) D = \Delta \Rightarrow (x+1)^2 D^2 \Delta (\Delta$

$= \Delta^2 - \Delta$ $\quad x = e^t$ in eq (A)

$\Rightarrow \left[\Delta^2 - \Delta - 3\Delta + 4\right] y = e^{2t}$

$\Rightarrow (\Delta^2 - 4\Delta + 4) y = e^{2t}$

$\Rightarrow (\Delta^2 - 4\Delta + 4) 2 = e^{2t}$

For yc we find the roots

$D^2 - 4\Delta + 4 = 0$

$\Delta^2 - 2\Delta - 2\Delta + 4 = 0$

$\Delta(\Delta - 2) - 2(\Delta - 2) = 0$

$D-2=0$, $D=2$
$D-2=0$, $D=2$
So the roots real and repeal, Thenexal solutions

$y=(c_1+c_2x)^{mx}$

$y=(c_1+c_3x)^{2x}$

for $yp=?$

$yp=\frac{1}{D^2-4D+4}$

$yp=\frac{2}{2D-4}e^{2t}$

If we put 2

$2D-4 \Rightarrow 2(2)-4=0$

we take again derivative

$yp=\frac{\cancel{2}}{\cancel{2}}e^{2t}$

$y=(c_1+c_2x)^{2t}e^{2t}$ → General Solution

Ans