Pg# 01

MID PAPER
DIFFERENTIAL EQUATION

I.D # 13131

Q1a) Define differential equation along with 2 examples?

DIFFERENTIAL EQUATION:-

A differential equation is an equation that relates the unknown function to some of its derivatives, which, of course, are not known either.

EXAMPLE OF DIFFERENTIAL EQUATION:-

$$\theta'' + \sqrt{\frac{g}{l}} \sin\theta = 0,$$

$$Rq' + \frac{1}{C} q = \sin\omega t,$$

$$P' = rP\left(1 - \frac{P}{K}\right);$$

$$T' = -h\,(T - Q).$$

b) Define a separable Differential (SD) equation?

SEPARABLE DIFFERENTIAL (SD) EQUATION:-

A differential equation having the form:

$$x' = f(x)\,g(t),$$

where the right side is the product of a function of a x & a function of t, is called a separable function equation.

i) Solve the following initial value problem (IVP) & find the interval of validity of the solution for SD equation

$$y' = \frac{xy^3}{\sqrt{1+x^2}} \quad y(0) = -1$$

SOLUTION:-

$$y' = \frac{xy^3}{\sqrt{1+x^2}}$$

$$\frac{1}{y^3}\frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}$$

$$\frac{1}{y^3}\,dy = \frac{x}{\sqrt{1+x^2}}\,dx$$

$$\int y^{-2}\,dy = \int \frac{x}{\sqrt{1+x^2}}\,dx$$

$$\int y^{-3}\,dy = \int x(1+x^2)^{-1/2}\,dx$$

$$\frac{y^{-3+1}}{-3+1} = \frac{1}{2}\int 2x(1+x^2)^{-1/2}\,dx$$

$$\frac{y^{-2}}{-2} = \frac{1}{2}\,\frac{(1+x^2)^{-1/2+1}}{-1/2+1} + C$$

$$\frac{-1}{2y^2} = \frac{1}{2}\,\frac{(1+x^2)^{1/2}}{1/2} + C$$

$$\frac{-1}{2y^2} = \cancel{2}\frac{1}{\cancel{2}}(1+x^2)^{1/2} + C$$

$$\frac{-1}{2y^2} = \sqrt{1+x^2} + C$$

$$2y^2 = \frac{-1}{\sqrt{1+x^2}} - \frac{1}{C}$$

$$y^2 = -1/2\,(1+x^2)^{-1/2} - 1/C$$

$$y = \frac{-1}{\sqrt{2}}\sqrt{(1+x^2)^{-1/2}} - \frac{1}{C}$$

$$y = \frac{-1}{\sqrt{2}}(1+x^2)^{1/2+1/2} - \frac{1}{C}$$

$$y = \frac{-1}{\sqrt{2}}(1) - \frac{1}{C}$$

$$y = \frac{-1}{\sqrt{2}} - \frac{1}{C} \rightarrow A$$

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Apply Initial Condition:-

$$y(0) = -1$$

$$-1 = \frac{-1}{\sqrt{2}} - \frac{1}{C}$$

$$\cancel{-}1 = \cancel{-}(\sqrt{2} + C)$$

$$1 = \sqrt{2} + C$$

$$1 = 2 + C$$

$$C = -1 \quad \text{Put in } (A)$$

$$y(x) = \frac{-1}{\sqrt{2}} - \frac{1}{-1}$$

$$\boxed{y(x) = \frac{-1}{\sqrt{2}} + 1}$$ Ans/=

ii) Solve the following for SD equations:

$$\frac{dx}{dt} = \frac{t}{x}$$

SOLUTION:-

$$\frac{dx}{dt} = \frac{t}{x}$$

Separating the variable

$$x\frac{dx}{dt} = t$$

Integrating

$$\int x \frac{dx}{dt}\, dt = \int t\, dt,$$

$$\int x\, dx = \int t\, dt$$

$$\boxed{\text{or, Since } dx = \frac{dx}{dt}\, dt,}$$

$$\frac{1}{2}x^2 = \frac{1}{2}t^2 + C$$

Q2 Solve the following IVP Linear differential method:

a) Explain the steps for solving Linear Differential ~~method~~ equation.

LINEAR DIFFERENTIAL EQUATION

STEPS:-

STEP 01:

multiply both sides of the normal form of the equation

$$x' + p(t)x = q(t)$$

by the integrating factor

$$\mu(t) = e^{\int p(t)\,dt} = e^{P(t)}$$

STEP 02:

OBTAIN

$$(e^{P(t)} x)' = e^{P(t)} q(t)$$

STEP 03:

$$e^{P(t)} x(t) = \int e^{P(t)} p(t)\,dt + C$$

STEP 04:

Multiply by $e^{-P(t)}$ to obtain the general solution.

$$x(t) = c^{-P(t)} \int e^{P(t)} p(t)\,dt + C e^{-P(t)}$$

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i) $\cos(x)y' + \sin(x)y = 2\cos^3(x) - 1$ $\quad y[\pi/4] = 3\sqrt{2}.$

$0 \le x \le \frac{\pi}{2}$

SOLUTION:-

$\cos(x)y' + \sin(x)y = 2\cos^3(x) - 1$

$y' + \frac{\sin(x)}{\cos(x)} y = \frac{2\cos^3(x)\sin(x) - 1}{\cos x}$

$y' + \tan(x) y = 2\cos^2(x)\sin(x) - \sec x \rightarrow (A)$

(1) Compare equation A with

$y' + P(x) y = q(x)$ we obtain

$P(x) = \tan(x)$, $q(x) = 2\cos^2(x)\sin(x)\ \sec x$

the integrated factor is

$M(t) = e^{\int P(x)dx}, = e^{\int \tan dx}, = e^{\int P(x)dx}, e^{-\ln(\cos x)}$

$= -\cos(x)$

(2) multiply $-\cos x$ to both side of eq (A)

$-\cos(x)y' - \tan(x)\cos(x)y = -2\cos^3(x)\sin x +$
$\sec(x)\cos(x)$

$-\cos(x)y' - \sin(x)y = -2\cos^3(x)\sin(x) + 1$

$(-\cos(x)y)' = -2\cos^3(x)\sin(x) + 1$

$$-\cos(x)\,y = -2\int \cos^3(x)\sin(x)\,dx + x + C$$

$$-\cos(x)\,y = 2\int \cos^3(x)(-\sin(x))\,dx + x + C$$

$$-\cos(x)\,y = 2\,\frac{\cos^{3+1}(x)}{3+1} + x + C$$

$$-\cos(x)\,y = \frac{1}{2}\cos^4(x) + x + C$$

$$y = -\frac{1}{2}\cos^3(x) - \frac{x}{\cos x} - \frac{C}{\cos x} \longrightarrow (1)$$

Apply initial condition

$$y\left[\frac{x}{y}\right] = 3\sqrt{2}$$

$$3\sqrt{2} = -\frac{1}{2}\cos^3\left(\frac{\pi}{4}\right) - \frac{\pi/4}{\cos(\pi/4)} - \frac{C}{\cos(\pi/4)}$$

$$3\sqrt{2} = -\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^3 - \frac{\pi/4}{(\sqrt{2}/2)} - \frac{C}{(\sqrt{2}/2)}$$

$$3\sqrt{2} = -\frac{1}{2}\left(\frac{2\sqrt{2}}{8}\right) - \frac{2\pi}{4\sqrt{2}} - \frac{2C}{\sqrt{2}}$$

$$3\sqrt{2} = \frac{-\sqrt{2}}{8} - \frac{\pi}{2\sqrt{2}} - \frac{2C}{\sqrt{2}}$$

$$6 = \frac{-2}{8} - \frac{\pi}{2} - 2C$$

$$6 = -\frac{1}{4} - \frac{\pi}{2} - 2C$$

$$6 + \frac{1}{4} - \frac{\pi}{2} = -2c$$

$$\frac{24 + 1 - 2\pi}{4} = -2c$$

$$\frac{25 - 2\pi}{4} = -2c$$

$$\frac{2\pi - 25}{8} = c$$

put in (1) equation

$$\boxed{y(x) = -\frac{1}{2}\cos^3(x) - \frac{x}{\cos^3} - \frac{(2\pi - 25)}{8\cos x}} \text{ Ans}$$

ii) $x' + 2x = \sin t$

SOLUTION:-

$$x' + 2x = \sin t$$

multiply DE by integrating factor

$$\mu(t) = e^{\int 2\,dt} = e^{2t}$$

$$(xe^{2t})' = e^{2t} \sin t.$$

integrated both sides.

$$xe^{2t} = \int e^{2t} \sin t \, dt + C$$

$$x(t) = e^{-2t} \int e^{2t} \sin t \, dt + Ce^{-2t}$$

$$x(t) = e^{-2t}\left[e^{2t}\left(\tfrac{2}{5} \sin t - \tfrac{1}{5} \cos t \right)\right] + Ce^{-2t}$$

$$= \frac{2}{5} \sin t - \tfrac{1}{5} \cos t + Ce^{-2t}$$

Q3) Solve the following IVP for the exact equation & find the interval of validity for the solution.

i) $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0, \; y(0) = -3$

SOLUTION:-

$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0, \; y(0) = -3$

$A = 2xy - 9x^2 \quad \& \quad A_y = 2x$

$B = 2y + x^2 + 1 \quad B_x = 2x$

$d_x = A$

$d_y = B$

$d = \int A\,dx$ OR $= \int B\,dy$

$d(x,y) = \int 2xy - 9x^2\,dx = x^2y - 3x^3 + h(y)$

$d_y = x^2 + h'(y) = 2y + x^2 + 1 = B$

$h'(y) = 2y + 1$

$h(y) = \int 2y + 1\,dy = y^2 + y + k$

$d(x,y) = x^2y - 3x^3 + y^2 + y + k = y^2 + (x^2+1)y - 3x^3 + k$

$y^2 + (x^2+1)y - 3x^3 + k = C$

$y^2 + (x^2+1)y - 3x^3 = C - k$

$y^2 + (x^2+1)y - 3x^3 = C$

$$(-3)^2+(0+1)(-3)-2(0)^3=C \qquad \boxed{C=6}$$

$$y^2+(x^2+1)y-3x^3-6=0$$

Now, as we in the seperable differential equation section, this is quadratic in y & so we can solve the y(x) by using quadratic formula.

$$y(x)=\frac{-(x^2+1)\pm\sqrt{(x^2+1)^2-4(1)(-3x^3-6)}}{2(1)}$$

$$=\frac{-(x^2+1)\pm\sqrt{x^4+12x^3+2x^2+25}}{2}$$

$$-3=y(0)=\frac{-1\pm\sqrt{25}}{2}$$

$$=\frac{-1\pm 5}{2}$$

$$=-3,2$$

$$y(x)=\frac{-(x^2+1)-\sqrt{x^4+12x^3+2x^2+25}}{2}$$

$$x^4+12x^3+2x^2+25=0$$

ii) $\frac{2ty}{t^2+4} - 2t - (2 - \ln(t^2+1))y' = 0 \quad y(5) = 0$

SOLUTION:-

$$\frac{2ty}{t^2+4} - 2t - (2 - \ln(t^2-1))y' = 0$$

$$= \frac{2ty}{t^2+1} - 2t + (\ln(t^2+1) - 2)y' = 0$$

$$A = \frac{2ty}{t^2+4} - 2t \quad , \quad A_y = \frac{2t}{t^2+1}$$

$$B = \ln(t^2+1) - 2 \quad , \quad B_y = \frac{2t}{t^2+1}$$

$$\alpha(t,y) = \int \frac{2ty}{t^2+1} - 2t \, dt = y\ln(t^2+1) - t^2 + h(y)$$

$$\alpha_y = \ln(t^2+1) + h'(y) = \ln(t^2+1) - 2 = B$$

$$h'(y) = (-2) \Rightarrow h(y) = -2y$$

$$\alpha(t,y = y\ln(t^2+1) - t^2 - 2y$$

$$y\ln(t^2+1) - t^2 - 2y = C$$

$$-25 = C$$

$$y(\ln(t^2+1) - 2) - t^2 = -25$$

$$y(t) = \frac{t^2 - 25}{\ln(t^2+1) - 2}$$

$$\ln(t^2+1) - 2 = 0$$

$$\ln(t^2+1) = 2$$

$$t^2+1 = e^2$$

$$t = \pm\sqrt{e^2-1}$$

Possible interval of validity

$$-\infty < t < -\sqrt{e^2-1}$$

$$-\sqrt{e^2-1} < t < \sqrt{e^2-1}$$

$$\boxed{\sqrt{e^2-1} < t < \infty}$$ Ans.