Date: Day: M T W T F S

(INU)

Name: Rizwan khan

ID: 17015

Semester: 3rd CS

Differential Equation

Mid Term

Q. Define differential equation along with 2 examples.

Differential Equation:

It is an equation with function and its one or more derivatives.

Example: (1)

$$y' = x^2 - 3$$

$$\Rightarrow \frac{dy}{dx} = (x^2 - 3)$$

$$\Rightarrow dy = (x^2 - 3)\,dx$$

Taking "$\int$" b.s

$$\Rightarrow \int dy = \int x^2\,dx - \int 3\,dx$$

$$y = \frac{x^3}{3} - 3x + k \quad (k \text{ is constant})$$

Example 2

$$\frac{dy}{dx} = 6 \qquad x=0,\ y=2$$

$$\Rightarrow dy = 6\,dx$$

$$\Rightarrow dy = 6\,dx$$

Taking '$\int$' on both sides

$$\int dy = \int 6\,dx$$

Date: Day: M T W T F S

$\Rightarrow$ $y = 6x + k$

By Applying bondary condition

$K = 2$

$x = 0$

$y = 2$

$y = 6x + 2$ Ans.

b) Define a seperable Differential Equation.

Seperable Differential Equation:

It any differential equation that we can write in the following form.

$$N(y)\frac{dy}{dx} = M(x)$$

(i) Solve the following IVP using SDE & find the interval of validity of the solution.

$$y' = \frac{xy^3}{\sqrt{1+x^2}} \qquad y(0) = -1$$

Solution:

$$y' = \frac{xy^3}{\sqrt{1+x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}}$$

$$dy\, y^{-3} = x(1+x^2)^{-1/2}\, dx$$

Integrate both sides

$$\int y^{-3}\,dy = \int x\,(1+x)^{-1/2}\,dx$$

$$\Rightarrow \quad \frac{y^{-2}}{-2} = \sqrt{1+x} + c$$

$$\Rightarrow \quad \frac{1}{-2y^2} = \sqrt{1+x} + c \quad \text{—(i)}$$

Put x=0 and y=-1

$$-\frac{1}{2(-1)^2} = \sqrt{1+(0)} + c$$

$$\Rightarrow \quad -\frac{1}{2} = \sqrt{1} + c$$

$$c = -\frac{3}{2}$$

Equation (i) becomes:

$$\frac{1}{-2y^2} = \sqrt{1+x}\ \cancel{+c} - \frac{3}{2}$$

By multiplying 2 on b/s

$$\frac{1}{y^2} = 3-2\sqrt{1-x^2}$$

$$y^2 = -\frac{1}{3-2\sqrt{1-x^2}}$$

Date:

Day: M T W T F S

Interval of validity

$$3-2\sqrt{1+x^2} > 0$$

$$\Rightarrow \quad 3 > 2\sqrt{1+x^2}$$

Taking square on b/s

$$9 > 4(1+x^2)$$

$$\Rightarrow \quad 9 > 4 + 4x^2$$

$$\Rightarrow \quad 9-4 > 4x^2$$

$$\Rightarrow \quad \frac{5}{4} > \frac{\cancel{4}x^2}{\cancel{4}}$$

$$\Rightarrow \quad \frac{5}{4} > x^2$$

Taking square root

$$\sqrt{\frac{5}{4}} > \sqrt{x^2}$$

$$\Rightarrow \quad -\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

Answer....

Date: Day: M T W T F S

(b) $y' = e^{-y}(2x-4) \quad y(5)=0$

Solution:

$$y' = e^{-y}(2x-4) \qquad y(5)=0$$

$$\frac{dy}{dx} = \frac{2x-4}{e^{+y}}$$

Multiying both sides by e^y

$$\frac{dy}{dx} \times e^y = \frac{2x-4}{\cancel{e^y}} \times \cancel{e^y}$$

$$\Rightarrow \frac{dy}{dx}(e^y) = 2x-4$$

Multiplying both sides by dx

$$\cancel{dx} \times \frac{dy}{\cancel{dx}} e^y = 2x-4 \, dx$$

$$dy \, e^y = 2x-4 \, dx$$

Taking integration on b/s

$$\int e^y dy = \int (2x-4)\, dx$$

$$e^y = x^2 + 4x + C$$

Natural log (ln)

$$y = \ln(x^2+4x+C)$$

Date:

Day: M T W T F S

Finding Value of constant 'C'

$x = 5$ and $y = 0$

$0 = \ln[(5)^2 - 4(5) + c]$

$0 = \ln(5 + c)$

$5 + c = 1$

$c = -5 + 1 \quad = c = -4$

Putting value of c

$y = \ln(x^2 - 4x - 4)$

Date: Day: M T W T F S

Q2 Solve the following IVP using linear differential equation.

i) Explain the steps for solving Linear diff equation.

Steps:

Here is step by step method for solving them:

1 **Step 1:**

Substitute $y = uv$ and $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

into

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

Step 2:

Factors the parts involving v.

Step 3:

Put the v term equal to zero.

Step 4:

Solve using seperable diff equation. to find u.

Step 5:

~~solve usi~~ substitute u back into equation we got at step 2

Step 6:

solve that to find v

Step 7:

Finally, substitute u and v into $y = uv$ to get our solution.

Date: Day: M T W T F S

(ii) $\cos(x)\,y' + \sin(x)\,y = 2\cot^2(x)\sin(x) - 1$

$\because\ y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$

$\because\ 0 \le x \le \frac{\pi}{2}$

Solution:

$$y' + \frac{\sin(x)}{\cos(x)}y = 2\cos^2(x)\sin(x) - \frac{1}{\cos x}$$

$$y' + \tan(x)\,y = 2\cos^2(x)\sin x - \sec x$$

$$\Rightarrow u(t) = e^{\int \tan(x)dx} = e^{\ln|\sec(x)|} = e^{\ln \sec x}$$

$$\Rightarrow \sec(x)\,y^3 + \sec(x)\tan(x)\,y = 2\sec(x)$$

$$\Rightarrow \cos^2 x\sin(x) - \sec^2(x)$$

$$\Rightarrow (\sec(x)\,y)^2 = 2\cos(x)\sin(x) - \sec^2(x)$$

$$\Rightarrow \int (\sec(x)\,y(x))'\,dx = \int 2\cos(x)\sin(x) - \sec^2(x)$$

$$\Rightarrow \sec x\,y(x) = -\frac{1}{2}\cos 2x - \tan x + c$$

$$\Rightarrow y(x) = -\frac{1}{2}(\cos x)(\cos 2x) - \cos x\tan x + \cos x$$

Date: Day: M T W T F S

$\Rightarrow$ $y(x) = -\frac{1}{2} \cos x \cos 2x - \sin(x) + C\cos x$

Put $y = 3\sqrt{2}$ & $x = \frac{\pi}{4}$

$3\sqrt{2} = y\left(\frac{\pi}{4}\right) = -\frac{1}{2}\cos\left(\frac{\pi}{4}\right)\cos\frac{\pi}{2} - \sin\frac{\pi}{2}$

$3\sqrt{2} = \frac{\sqrt{2}}{2} + C\frac{\sqrt{2}}{2}$

$C = 7$

It becomes.

$y(x) = -\frac{1}{2} \cos(x)\cos(2x - \sin x + 7\cos x$

(iii) $x' + 2x = \sin t$

Solution:

? ?

Date: Day: M T W T F S

Q3 Solve the following IVP for exact equation &
(i) find interval of validity for the solution

(i) $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0 \quad y(0) = -3$

Solution:

$M = 2xy - 9x^2 \qquad My = 2x$

$N = 2y + x^2 \qquad Nx = 2x$

Now for finding $\psi(x,y)$?

$\psi x = M$

$\psi y = N$

$\psi = \int M dx \quad$ or $\quad \psi = \int N dy$

$\psi y = x^2 + h'(y) = 2y + x^2 + 1 = N$

$h'(y) = 2y + 1$

$h(y) = \int 2(y) + 1 dy = y^2 + y + k.$

$\psi(x,y) = x^2 y - 3x^2 + y^2 + y + k$

$\Rightarrow y^2 + (x^2 + 1) y - 3x^2 + k = c$

$\Rightarrow y^2 + (x^2 + 1) y - 3x^2 = c - k$

$y^2 + (x^2 + 1) y - 3x^2 = c$

So Putting values of x and y

$(-3)^2 + (0+1)(-3) - 3(0)^3 = c$

$c = 6$

$y^2 + (x^2 + 1) y - 3x^2 - 6 = 0$

Quadratic formula

$$y(x) = \frac{-(x^2+1) \pm \sqrt{(x^2+1)^2 - 4(1)(-3x^2-6)}}{2(1)}$$

$$= \frac{-(x^2+1) \pm \sqrt{x^2 + 12x^3 + 2x^2 + 25}}{2}$$

$y(x) = -(x^2+1) - \sqrt{x^4 + 12x^2 + 25}$

Date: Day: M T W T F S

(ii) $\frac{2+y}{t^2+1} - 2t - (2 - \ln(t^2+1))y' = 0 \because y(5) = 0$

Solution:

$M = \frac{2+y}{t^2+1} - 2t \qquad M_y = \frac{2t}{t^2+1}$

$N = \ln(t^2+1) - 2 \qquad N_t = \frac{2t}{t^2+1}$

$\int$ of 'M'

$\psi(x,y) = \int \frac{2+y}{t^2+1} - 2t \; dy = y\ln(t^2+1) - t + h(y)$

"Differentiate"

$\psi_y = \ln(t^2+1)\, h'(y) \; \ln(t^2+1) - 2$

$h'(y) - 2 \Rightarrow h(y) = -2y$

$\psi(t,y) = y \cdot \ln(t^2+1) - t^2 - 2y$

$y\ln(t^2+1) - t^2 - 2y = c$

By putting values we get c

$y\ln(t^2+1) - 2) - t^2 = -25$

$y(t) = \frac{t^2 - 25}{\ln(t^2+1) - 2}$

$\ln(t^2+1) - 2 = 0$

$\ln(t^2+1) = 2$

$t^2 + 1 = e^2$

$t = \pm\sqrt{e^2 - 1}$ Ans.