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A First Course in Differential Equations Third Edition

March 2, 2015

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

To David Russell Logan

Contents

Preface to the Third Edition

This new edition remains in step with the goals of earlier editions, namely, to offer a concise treatment of basic topics covered in a post-calculus differential equations course. It is written for students in engineering, biosciences, physics, economics, and mathematics. As such, the text is strongly guided by applications in those areas.

The last twenty-five years witnessed dramatic changes in basic calculus courses and in differential equations. One driver of change has been the availability of technology and its role in a standard course, and another is the level of preparation of students with regard to their ability to perform analytical manipulations. Writing a text for such a diverse audience poses a substantial challenge. Some students need only know what a differential equation means and what it implies qualitatively to understand concepts in their areas; others, who plan on taking advanced courses in engineering or the physical sciences where the mathematics is more intense, require ability to perform analytic calculations. This text makes an effort to balance these two issues.

Some outstanding textbooks have been written for this course. But many are calculus-like and voluminous, with extensive graphics, marginal notes, and numerous examples and exercises; they cover many, many more topics than can be discussed in a one-semester course. I have often felt that students become overwhelmed, distracted, and even insecure about skipping material and jumping around in a text of several hundred pages.

An overarching philosophy in this text is that *you can't cover everything.* Therefore, it is more concise and written in a plain, user friendly format that is accessible to science and engineering students. Often these students have limited time and they appreciate a smaller parcel where it is clear what they should know. One success of the text has been that it gives instructors who want this type of coverage an alternative to existing texts. Another characteristic is that it encourages students to begin developing their analytical thinking for future studies; this includes some formula manipulation and understanding derivations. Students should slowly advance in their ability to read mathematics in preparation for more advanced, upper level texts in their areas, which require a lot of the reader.

The topics are standard and the Table of Contents lists them in detail. Briefly, the chapters cover:

- **Chapter 1. First-order equations.** Separable, linear, and autonomous equations; equilibrium solutions, stability and bifurcation. Other special types of equations, for example, Bernoulli, exact, and homogeneous equations, are covered in the Exercises with generous guidance. Many applications are discussed from science, engineering, economics, and biology.
- **Chapter 2. Second-order linear equations.** The emphasis is on equations with constant coefficients, both homogeneous and nonhomogeneous, with most examples being spring-mass oscillators and electrical circuits. Other than Cauchy–Euler equations, variable coefficient equations are not examined in detail. There are three optional sections covering reduction of order, higher-order equations, and steady-state heat transfer, which deals with simple boundary value problems.
- **Chapter 3. Laplace transforms.** The treatment is standard, but without overemphasizing partial fraction decompositions for inversion. Use of the enclosed table of transforms is encouraged. This chapter can be covered at any time after Chapter 2.
- **Chapter 4. Linear systems.** This chapter deals only with two-dimensional, or planar, systems. It begins with a discussion of equivalence of linear systems and second-order equations. Linear algebra is kept at a minimum level, with a very short introductory section on notation using vectors and matrices. General solutions are derived using eigenvalues and eigenvectors, and there are applications to chemical reactors (compartmental analysis), circuits, and other topics. There is a thorough introduction to phase plane analysis and simple geometric methods.

- **Chapter 5. Nonlinear systems.** This chapter revolves around applications, e.g., classical dynamics, circuits, epidemics, population ecology, chemical kinetics, malaria, and more. Typically, inclusion of this chapter requires a 4-credit semester course.
- **Chapter 6. Computation of solutions.** This brief chapter first discusses the Picard iteration method, and then numerical methods. The latter include the Euler and modified Euler methods, and the Runge–Kutta method. All or parts of this chapter can be covered or referred to at any time during the course.

A standard 3-credit semester course can be based on Chapter 1 through most of Chapter 4. A 4-credit course can include topics from Chapter 5 on nonlinear systems.

This edition of the text incorporates many changes. Some topics have been rewritten and rearranged. I made the effort to introduce an easier-to-read format and highlight important concepts. There is a increase in the number of routine examples and exercises. A major notational change is that generic functions in differential equations, previously represented by $u = u(t)$, have been changed to the more common $x = x(t)$. The number and variety of applications is substantially increased, and several exercises throughout the book have enough substance to serve as mini-projects for students. Starred sections (*) are optional. Time availability in a one-semester course was an overriding factor, and some topics, such as power series and special functions, are not covered.

Two appendices complement the chapters. There is a new appendix *Review and Exercises* that concisely summarizes methods from Chapters 1 and 2, supplemented with several exercises, along with solutions. It also includes a set of chapter exercises on which students can test their skills. Instructors can use the exercises as a test bank. A second appendix is a MATLAB® supplement that summarizes MATLAB commands and demonstrates simple code writing, as well as use of its built-in programs and symbolic packages to solve problems in differential equations. MATLAB is *not required* for the text. Rather, students are encouraged to use the software available to them. Many exercises can be done with an advanced scientific calculator. Solutions to the even-numbered problems can be found at http://www.springer.com/ and on the author's web site.

Several individuals deserve my heartfelt acknowledgments. User's suggestions have become part of this revision, and I greatly appreciate their interest in making it a better text. Also I thank my many students who, over the last several years, have endured my lectures and exams and have generously given me valuable advice; very often they reminded me who my audience was. My son

David, to whom I dedicate this book, was a frequent and meticulous grader who always advocated for students and often altered my own perspective in teaching undergraduates. Elizabeth Loew, my editor at Springer, deserves special recognition for her continuous attentiveness to the project and her expert support. I have found Springer to be an extraordinary partner in this project.

Corrections, comments, and suggestions on the text are greatly appreciated. Contact information is on my web site: www.math.unl.edu/~jlogan1. Additional material, including an errata, will be posted there.

J. David Logan
Willa Cather Professor

1
First-Order Differential Equations

Readers are familiar with solving algebraic equations. For example, the solutions to the quadratic equation

$$x^2 - x = 0$$

are easily found to be $x = 0$ and $x = 1$, which are *numbers*. An ordinary differential equation, or just *differential equation*, is another type of equation where the unknown is not a number, but a *function*. We call the unknown function $x(t)$ and think of it as a function of time t. Simply, a differential equation is an equation that relates the unknown function to some of its derivatives, which, of course, are not known either.

Why are differential equations so important that they deserve an entire course, or even lifetime, of study? Well, differential equations arise naturally as *mathematical models* in areas of science, engineering, economics, and many other subjects. Physical systems, biological systems, and economic systems, and so on are marked by change, or *dynamics*. Differential equations describe those dynamical changes. The unknown function $x(t)$, often called the *state function*, could be the distance along a line, the current in an electrical circuit, the concentration of a chemical undergoing reaction, the population of an animal species in an ecosystem, or the demand for a commodity in a micro-economy. Differential equations represent *laws* that govern change, and the unknown state $x(t)$, for which we solve, describes how the changes occur. The bottom line is that most laws of nature, and other systems, relate the rate, or derivative, at which some quantity changes to the quantity itself—thus a differential equation.

In this text we set up and solve differential equations. Often the solution to an equation is given by a general formula, and some may want to memorize the formulas. Students should realize that this strategy is inappropriate; it leads to little understanding and memorized formulas fade quickly from memory. What is important in this subject is conceptual, that is, the *understanding* of its origin and the *process* of solving the equation and interpreting the results.

Historically, differential equations dates to the mid-seventeenth century when the calculus was developed by Isaac Newton (c. 1665) in the context of studying the laws of mechanics and the motion of planets (published in his *Principia*, 1687). In fact, some might say that calculus was invented to describe how objects move. Other famous mathematicians and scientists of that era, for example, Leibniz, the Bernoullis, Euler, Lagrange, and Laplace contributed important results into the 1700s, and Cauchy developed some of the first theoretical concepts in the 1800s. Differential equations is now the principal tool for applications in all areas of mechanics, thermodynamics, electromagnetic theory, quantum theory, and so on. It continues today with the study of dynamical systems and nonlinear phenomena in biology, chemistry, economics, and almost every area where the dynamics of systems is important. By studying differential equations, students are rewarded with a knowledge of one of the monuments of mathematics and science, and they see the great connection between nature and mathematics like they may never have imagined.

1.1 First-Order Equations

1.1.1 Notation and Terminology

We begin with a simple example from elementary physics. Suppose a body of mass m moves along a line with constant velocity V. Suddenly, say at time $t = 0$, an external resistive force F acts on the body given by $F = -kv(t)$, where $k > 0$ is a fixed constant called the drag coefficient. So the force is proportional to the velocity for all times $t > 0$. Intuitively, the body will slow down and its velocity will decrease. From this information we can predict the velocity $v(t)$ of the body at any time $t > 0$. Newton's second law of motion states that the mass of the body times its acceleration equals the force upon it, or $ma = F$. We also know from calculus that the derivative of velocity is acceleration, or $a = v'(t)$. (We are using the *prime* notation for derivative.) Therefore, Newton's law implies

$$mv'(t) = -kv(t).$$

This is an example of a differential equation. The unknown is the velocity function $v = v(t)$. We call it the *equation of motion* of the system. It involves an unknown function $v(t)$, the velocity, and its derivative $v'(t)$. If we can find a function $v = v(t)$ that "works" in the equation and also satisfies $v(0) = V$, which is the initial condition on the velocity, then we will have determined the velocity of the particle at any time and solved the differential equation. In summary, we wish to solve for the velocity $v = v(t)$ in the problem

$$mv'(t) = -kv(t), \tag{1.1}$$

$$v(0) = V, \tag{1.2}$$

where m, k, and V are fixed parameters, or constants.

After some practice we will be able to easily solve the equation and determine that the velocity decreases exponentially, or

$$v(t) = Ve^{-kt/m}, \ t \geq 0.$$

This formula for $v(t)$ is the *solution* to the problem (1.1)–(1.2). To check that it works we substitute this expression into (1.1) and (1.2).

$$mv'(t) = mV\left(-\frac{k}{m}\right)e^{-kt/m} = -k\left(Ve^{-kt/m}\right) = -kv(t).$$

Moreover, substituting $t = 0$, we find $v(0) = V$. So it checks.

To review, the differential equation (1.1) *governs* the dynamics of the body. We set it up using Newton's second law, and it contains the unknown function $v(t)$, along with its derivative $v'(t)$. The solution $v(t)$ predicts how the system evolves in time. We can sketch a generic graph of the solution as a visual representation of the motion. See Figure 1.1.

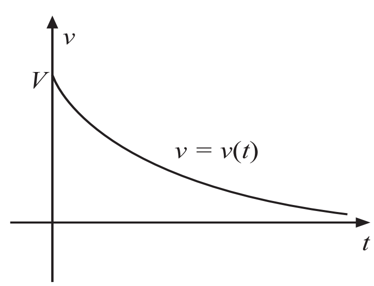

Figure 1.1 A generic plot of the velocity $v = v(t)$, a decreasing exponential, and the solution of (1.1)–(1.2).

In the previous example we used $v = v(t)$ as the unknown velocity. But if the unknown is "population", then we may use $p = p(t)$ or $N = N(t)$; if the unknown is current in a circuit, we may use $I = I(t)$. To discuss differential equations in a generic setting with no specific context, we use $x = x(t)$ as the unknown function. We think of x as the dependent variable and t, which is time, as the independent variable. Derivatives are denoted by a prime or by the Leibniz notation,

$$x' \quad \text{or} \quad \frac{dx}{dt},$$

and so on for higher derivatives. We use these notations interchangeably. Recall that the first derivative of a quantity is the "rate of change of the quantity," measuring how fast the quantity is changing, and the second derivative measures how fast the rate is changing. For example, if the state of a mechanical system is position, then its first derivative is velocity and its second derivative is acceleration, or the rate of change of velocity.

A **differential equation** is an equation that relates the state of a system $x(t)$ to some of its rates of change, as expressed by its derivatives $x'(t), x''(t), ...,$ and so on. In different words, *it is an equation that describes how a state $x(t)$ of a system changes in time*. The common strategy in science, engineering, economics, and so on, is to formulate basic principles in terms of differential equations for the unknown state $x = x(t)$ and then solve the equation to find the state, thus determining how the system evolves in time.

Because the time variable is understood, we often drop the time dependence notation in a differential equation and write, for example, the differential equation (1.1) as

$$mv' = -kv \quad \text{or} \quad m\frac{dv}{dt} = -kv. \qquad \Box$$

Example 1.1

Here are four examples of differential equations that arise in various applications:

$$\theta'' + \sqrt{\frac{g}{l}}\sin\theta = 0, \qquad \text{(pendulum equation)}$$

$$Rq' + \frac{1}{C}q = \sin\omega t, \qquad \text{(RC circuit equation)}$$

$$p' = rp\left(1 - \frac{p}{K}\right), \qquad \text{(population equation)}$$

$$T' = -h(T - Q). \qquad \text{(heating-cooling equation)}$$

The first equation models the angular deflections $\theta = \theta(t)$ of a pendulum of length l; the second models the charge $q = q(t)$ on a capacitor in an electrical circuit containing a resistor and a capacitor, where the current is driven by an electromotive force $\sin \omega t$ of frequency ω; in the third equation, called the logistic equation, the unknown function $p = p(t)$ represents the population of an animal species in a closed environment; r is the population growth rate and K represents the capacity of the environment to support the population; the last is Newton's law of cooling, which models of temperature $T = T(t)$ of an object placed in an environment of Q degrees; h is the heat loss coefficient. The unspecified constants in the various equations, l, R, C, ω, r, K, h, and Q are called **parameters**, and they can take any value we choose. Most differential equations that model physical processes contain such parameters. The constant g in the pendulum equation is a **fixed parameter** representing the acceleration of gravity on earth. In mks units, $g = 9.8$ meters per second squared. Because time-dependence is understood, in the first equation θ means $\theta(t)$ and θ'' means $\theta''(t)$, and so on. □

The **order** of a differential equation is the order of the highest derivative appearing in the equation. For example, $x' + 2x = t$ is a first-order equation, and $x'' = -x' - 7x$ is a second-order equation. In Example 1.1, the first is second-order, and the the other three are first-order.

To formulate general principles, we write a generic first-order equation for an unknown function $x = x(t)$ as

$$x' = f(t, x), \tag{1.3}$$

where f represents some given functional relationship between t and x. This form is called the **normal form** of a first-order equation. A function $x = x(t)$ is a **solution**[1] of (1.3) on a time interval $I : a < t < b$ if it is differentiable on I and, when substituted into the equation, it satisfies the equation identically for every $t \in I$; that is,

$$x'(t) = f(t, x(t)), \text{ for every } t \in I.$$

"Satisfies identically" means "can be reduced to $0 = 0$." *To check if we have a solution, we merely substitute the function in question into the differential equation and check that it reduces to an identity.*

The problem of solving a differential equation with unknown $x = x(t)$, subject to a condition $x(t_0) = x_0$, that is,

$$x' = f(t, x),$$
$$x(t_0) = x_0,$$

[1] We are overburdening the notation by using the same symbol x to denote both a variable and a function. It would be more precise to write "$x = \varphi(t)$ is a solution," but we choose to stick to the common use, and abuse, of a single letter.

is called an **initial value problem** (**IVP**). Here, t_0 is a fixed value of time and x_0 is a fixed value of x, and $x(t_0) = x_0$ is called an **initial condition**. Geometrically, an initial value problem asks what solution curve $x = x(t)$ plotted in the tx plane passes through the fixed point (t_0, x_0). An example of an initial value problem is the problem (1.1)–(1.2) discussed at the beginning of this section. Later in Chapter 1 we state some very general conditions that guarantee that this problem has a solution. Finally, the **interval of existence** of an IVP is the largest time interval where the solution is valid.

Example 1.2

Consider the two similar IVPs

$$x' = 1 - x^2, \quad x(0) = 0,$$
$$x' = 1 + x^2, \quad x(0) = 0.$$

The first has solution

$$x(t) = \frac{e^{2t} - 1}{e^{2t} + 1},$$

which exists for every value of t; the interval of existence is $-\infty < t < \infty$. Yet the second has solution

$$x(t) = \tan t,$$

existing only on the interval $-\pi/2 < t < \pi/2$, which is its interval of existence. (Any other branch of the tangent function will not pass through the point $t = 0,\ x = 0$.) Therefore, solutions to IVPs need not be valid for all times t. □

1.1.2 Growth–Decay Models

In this section we present a very simple model of *decay*. Not only is the model an important application, but we use it here to introduce key ideas and terminology. We use these ideas and terms in almost every example in this book.

Many processes in nature can be described as *decay* processes. For example, radioactive materials decay, (^{14}C, for example, in carbon dating), insect populations decay (mortality), chemicals released in the ground degrade over time, and drugs given by injection in the blood decay. These processes are all described by the decay equation

$$x' = -rx, \qquad \text{(decay equation)} \tag{1.4}$$

where $r > 0$ is the **decay rate**. Here, $x = x(t)$, the state, represents some quantity of interest. In words, the rate of change of the quantity is proportional to the amount present; $-r$ denotes the proportionality constant and in the following discussion we assume it is a fixed number. Although it is easy to guess a solution to this equation, we want to solve it from basic principles. We rewrite the equation as

$$\frac{x'}{x} = -r,$$

and then observe from the chain rule that

$$\frac{x'}{x} = \frac{d}{dt}\ln x.$$

Thus the equation becomes

$$\frac{d}{dt}\ln x = -r.$$

Remembering that x is a function of t, we integrate both sides with respect to t to get

$$\int \frac{d}{dt}\ln x\, dt = -\int r\, dt.$$

The integral of the derivative of a function is the function itself (integration and differentiation are inverse processes), and so we get

$$\ln x = -rt + C_1,$$

where C_1 is an arbitrary constant of integration. Solving for x gives

$$x = e^{-rt+C_1} = e^{C_1}e^{-rt} = Ce^{-rt},$$

where C is an **arbitrary constant** which can take on any value. (Note: if C_1 is an arbitrary constant, then $C = e^{C_1}$ must also be arbitrary.) Therefore, there are *infinitely many solutions* to the decay equation (1.4) given by

$$x(t) = Ce^{-rt}. \tag{1.5}$$

The infinite set of solutions (1.5) of a first-order equation is called the **general solution** of the equation. The general solution contains an arbitrary constant C, and as such it is also referred to as a **one-parameter family** of solutions—one for each choice of C. They are also referred to as the **integral curves** of the differential equation. These curves are plotted in Figure 1.2. One can ask how there can be infinitely many solutions to a decay problem. In a real system there is typically an **initial condition** imposed on the state $x(t)$; that is,

$$x(0) = x_0, \qquad \text{(initial condition)}$$

where x_0 is a prescribed, fixed state at time $t = 0$. This initial condition picks out a specific value of the arbitrary constant C, and hence a specific solution

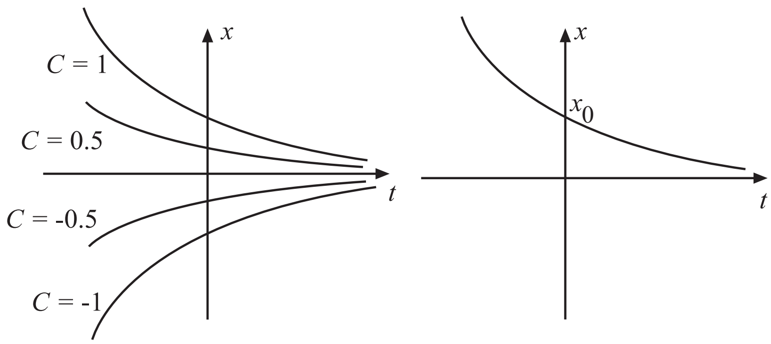

Figure 1.2 (Left) Plots of four integral curves, or solution curves (1.5), of the differential equation (1.4), for four values of C. (Right) A particular solution satisfying the initial condition $x(0) = x_0$.

curve. In this case, $x(0) = x_0 = C\exp(-r \cdot 0) = C$. Therefore we have selected out a **particular solution**

$$x(t) = x_0 e^{-rt}$$

of the DE (1.4). To repeat, of the many solutions, we have chosen the one that satisfies the initial condition. As we mentioned earlier, the problem of solving a differential equation subject to an initial condition is called an **initial value problem**. In the preceding discussion, the initial condition is given at $t = 0$, but in general it may be given at any fixed time $t = t_0$.

Example 1.3

(**Radioactive decay**) Four grams of a sample of a radioisotope decays to 0.8 grams in 3 years. What is the decay constant r? What is the *half-life*, or the time when only half of the sample remains? If $x = x(t)$ is the amount, in grams, of the isotope, the decay law is

$$x' = -rx, \quad x(0) = 4.$$

The general solution of the differential equation is

$$x(t) = Ce^{-rt}.$$

The initial condition $x(0) = 4$ implies $x(0) = Ce^{-r(0)} = C = 4$. Therefore, we have

$$x(t) = 4e^{-rt}.$$

After 3 years we have $x(3) = 4e^{-3r} = 0.8$. Solving for r gives

$$r = -\frac{1}{3}\ln\left(\frac{0.8}{4}\right) = 0.536 \text{ per year},$$

which is the decay rate. To find the half life τ we set

$$2 = 4e^{-0.536\tau}.$$

Solving for τ gives $\tau = -(1/0.536)\ln 0.5 = 1.293$ years. □

Example 1.4

(**Population growth**) Ecology is the study of how organisms interact with their environment. A fundamental problem in population ecology is to determine what mechanisms operate to regulate animal populations. For the human population $p = p(t)$, Thomas Malthus, an essayist in the late 1700s, proposed the model, or law,

$$\frac{dp}{dt} = rp, \tag{1.6}$$

which states in words that the population growth rate p' is proportional to the current population p. The proportionality constant r, given in dimensions of time^{-1} (per time), is called the *growth rate*. We can regard r as depending on births and deaths in the population; for example, $r = b - m$, where b is the birth rate and m is the mortality rate. This equation is solved exactly like the decay equation. The only difference is a minus sign. The general solution is

$$p(t) = Ce^{rt},$$

where C is an arbitrary constant. If $p(0) = P_0$ is the initial population (where $t = 0$ is the time we begin counting), then $C = P_0$ and the population grows exponentially according to

$$p(t) = P_0 e^{rt}.$$

As an aside, note that this is exactly the same law as for growth of money compounded continuously at interest rate r. □

EXERCISES

1. Verify by direct substitution that the given differential equation has the solution as indicated.

   a) $x' = \frac{2x}{t}, \quad x = t^2.$ b) $x' = -\frac{t}{x}, \quad x = \sqrt{6 - t^2}.$

2. Verify the solutions to the initial value problems in Example 1.2.

3. Which of the following functions,

$$x(t) = \frac{1}{t}, \quad x(t) = \frac{2}{t}, \quad x(t) = \frac{1}{t-2},$$

   is a solution to the DE $x' = -x^2$?

4. Show that both

$$x_1(t) = e^{-t}\cos t \quad \text{and} \quad x_2(t) = e^{-t}\sin t$$

   are solutions to the second-order differential equation

$$x'' + 2x' + 2x = 0.$$

   Show that

$$x(t) = Ae^{-t}\sin t + Be^{-t}\cos t$$

   is a solution for any values of the constants A and B.

5. Show that $x(t) = \ln(t + C)$ is a one-parameter family of solutions, or integral curves, of $x' = e^{-x}$, where C is an arbitrary constant. Plot the integral curves using the values of C given by $C = -2, -1, 0, 1, 2$. On the plot indicate the particular solution that satisfying $x(0) = 0$.

6. Find a solution $x = x(t)$ of the equation $x' + 2x = t^2 + 4t + 7$ in the form of a quadratic function of t, that is, of the form $x(t) = at^2 + bt + c$, where a, b, and c are to be determined.

7. Find values of m for which $x(t) = t^m$ is a solution to $2tx' = x$.

8. Find values of m for which $x(t) = t^m$ is a solution to $t^2x'' - 6x = 0$.

9. Find two values of λ for which $x(t) = e^{\lambda t}$ is a solution of the differential equation $2x'' - 5x' - 3x = 0$.

10. Show that the one-parameter family of straight lines $x(t) = Ct + f(C)$ is a solution to the differential equation $tx' - x + f(x') = 0$ for any value of the constant C.

11. Plot the one-parameter family of curves $x(t) = (t^2 - C)e^{3t}$ for different values of C. Find a differential equation whose solution is $x = x(t)$. Hint: Find x' and then obtain a relation between t, x, and x'.

12. (*Physics*) In deep water, the intensity of light $I = I(x)$ at a depth x meters below the water surface is modeled by the equation $I' = -1.4I$. At what depth is the light intensity 1% that at the surface? (Observe that x is the independent variable in this exercise.)

13. (*Carbon dating*) The half-life of ^{14}C (Carbon-14) is 5730 years. That is, it takes this many years for half of a sample of ^{14}C to decay. If the decay of ^{14}C is modeled by $x' = -rx$, where x is the amount of ^{14}C, find the decay constant r. (Answer: $r = 0.000121\ \text{yr}^{-1}$). In an artifact the percentage of the original ^{14}C remaining at the present day was measured to be 20 %. How old is the artifact?

14. (*Carbon dating*) In 1950, charcoal from the Lascaux Cave in France gave an average count of 0.09 disintegrations of ^{14}C (per minute per gram). Living wood gives 6.68 disintegrations. Estimate the date that individuals lived in the cave. Note: The amount of ^{14}C is often measured in disintegrations per minute per gram.

15. (*Radioactivity*) The half-life of radioactive cobalt is 5.3 years. After a nuclear reactor accident, the surrounding region had 100 times the amount of cobalt acceptable for habitation. How long will it be before the region is habitable?

16. (*Growth*) The U.S. Census estimated that the world population was 6 billion in 1999, and it was increasing 212,000 per day. What is the annual growth rate? At this rate, what is the predicted population of the world in 2050?

17. (*Mortality*) An insect population dies off exponentially and is governed by the equation $p' = -\mu p$, where μ is the mortality rate. If 1000 insects hatch, and only 90 remain after 2 days, what is the mortality rate?

1.1.3 Geometric Approach

What does a differential equation $x' = f(t, x)$ tell us geometrically? At each fixed point (t, x) of the tx plane, the value of $f(t, x)$ is the slope x' of the solution curve $x = x(t)$ that passes through that point. This is because

$$x'(t) = f(t, x(t)).$$

This simple fact suggests a useful graphical method for constructing approximate solution curves for a first-order differential equation without finding the solution. Through each point of a selected set of points (t, x) in some region (or window) of the tx plane we draw a short line segment (dash) with slope $f(t, x)$. The collection of all these line segments, or mini-tangents, forms the **direction field**, or **slope field**, for the equation. We may then roughly sketch solution curves that fit into this direction field; the curves have the property that at each point the tangent line has the same slope as the slope of the direction field.

Example 1.5

The slope field for the differential equation $x' = -x + 2t$ is defined by the right side of the differential equation, $f(t, x) = -x + 2t$. The slope field at the point $(2, 4)$ is $f(2, 4) = -4 + 2 \cdot 2 = 0$. This means the solution curve that passes through the point $(2, 4)$ has slope 0. Because it is tedious to calculate several mini-tangents, simple programs have been developed for advanced calculators and computer algebra systems that perform this task automatically. Figure 1.3 shows a slope field and several solution curves that have been fit into the field. (The figure was easily created using MATLAB.) □

In most problems we do not need the detail shown in Figure 1.3; it is sufficient to know how to calculate the slope field at a few specially selected points (t, x). This gives the direction of the solution curves through those points. In particular, it is useful to find the set of all points in the plane where the slope field, or derivative, is either positive or negative and where it is zero (horizontal).

The sets of points where the slope field is *zero* are called **nullclines**; these are especially of interest in problems because they indicate where the slope may change signs. Formally, the *nullclines* of the differential equation $x' = f(t, x)$ is the set of points (t, x) for which $f(t, x) = 0$. The nullclines usually form a set of curves in the plane. In the regions between the nullclines, the slope field is positive or negative; it is often sufficient to select a point in each of those regions and calculate which. One alternative is to sketch the set of points (t, x) where the slope field has a constant value, that is, $f(t, x) = k$, for some fixed k. These curves are called **isoclines**.

Example 1.6

Consider the differential equation

$$x' = -tx + x^2 = x(x - t).$$

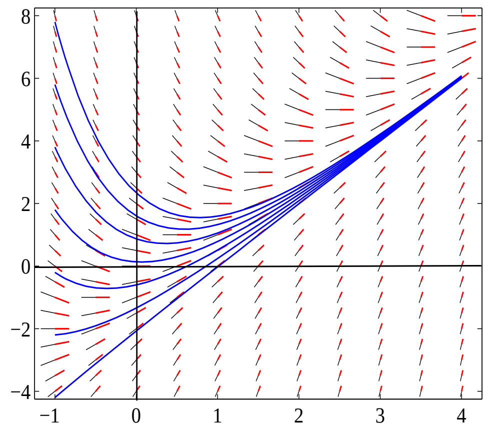

Figure 1.3 The slope field for the equation $x' = -x + 2t$ in the window $-1 \le t \le 4$, $-4 \le x \le 8$, shown with several solution curves. If an initial condition $x(t_0) = x_0$ is imposed, then *one* solution curve is selected, the one passing through the point (t_0, x_0).

The nullclines, found by setting $x' = 0$, are $x = 0$ (the t axis) and the diagonal line $x = t$ (both shown dashed in Figure 1.4). We note that $x' > 0$ when $x > 0$ and $x > t$, or $x < 0$ and $x < t$. And, $x' < 0$ when $x < 0$ and $x > t$, or $x > 0$ and $x < t$. Slopes are sketched on the plot in the appropriate four regions separated by the nullclines. Using the directions we can sketch approximate solution curves as shown in Figure 1.4. The nullcline $x = 0$ indicates a line of horizontal tangents and in fact indicates a constant solution $x(t) = 0$ to the differential equation. □

Example 1.7

The differential equation

$$x' = x(x^2 - t)$$

has nullclines $x = 0$ (the t-axis) and the parabolic curve $x^2 = t$, or $x = +\sqrt{t}$, $x = -\sqrt{t}$. Note that, as in the last example, $x(t) = 0$ is a solution to the equation. Where is the slope field positive? Negative? □

A problem in differential equations is just opposite of that in differential calculus. In calculus we know the function (curves) and are asked to find the

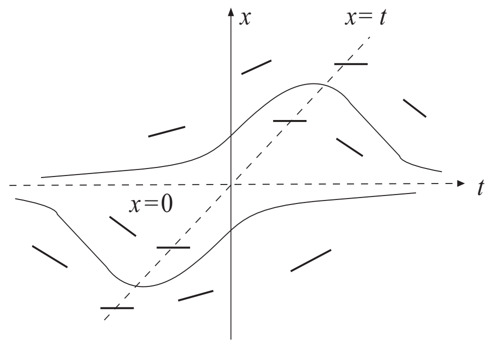

Figure 1.4 The slope field and nullclines $x = 0$, $x = t$ (dashed) for the differential equation $x' = -tx + x^2$, with two approximate solution curves fit into the field. Note that the nullcline $x = 0$ is a solution to the DE, but $x = t$ is not.

derivative (slopes); in differential equations we know the slopes and try to find the functions, or curves, that fit the slopes.

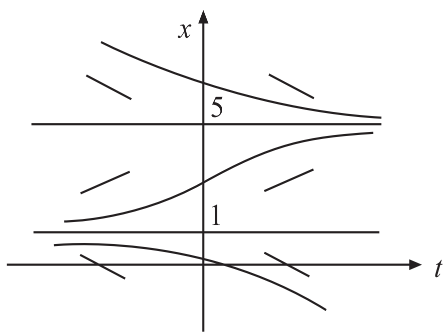

Figure 1.5 The two nullclines $x = 1$, $x = 5$ for the equation $x' = 2(x - 1)(5 - x)$, both of which are constant solutions. Between the nullclines the slopes are positive and negative where indicated; approximate solution curves are indicated.

There is simplicity in the slope field for a differential equation that has no explicit dependence on time, that is,

$$x' = f(x).$$

On each fixed horizontal line in the tx plane, where x has the same value, the slope field has the same value on that line. Moreover, the values of x where

$f(x) = 0$ are constant solutions to the differential equation.

Example 1.8

Refer to Figure 1.5. Along the line $x = 2$ the equation $x' = 2(x-1)(5-x)$ the slope field has value 6; so, every solution curve crossing that line has slope 6; this is an isocline. Along $x = 3$ the slope is 8, another isocline. For equations of this type, there are horizontal nullclines where $x' = f(x) = 0$. Here, the nullclines are $x = 1$ and $x = 5$. It is important to notice that these horizontal nullclines define constant solutions $x(t) = 1$ and $x(t) = 5$ to the differential equation. These important solutions are called **equilibrium solutions**, or steady-states, because they do not change in time. Above the line $x = 5$ and below the line $x = 1$ the slope field is negative, while in the region $1 < x < 5$ the slope field is positive. We can sketch in a few approximate solution curves, as shown in Figure 1.5. This topic is discussed in detail in Section 1.5. □

EXERCISES

1. By hand, sketch the slope field for the DE $x' = x(1 - x/4)$ in the window $0 \leq t \leq 8$, $0 \leq x \leq 8$ at the integer lattice points. What is the value of the slope field along the lines $x = 0$ and $x = 4$? Show that $x(t) = 0$ and $x(t) = 4$ are constant solutions to the DE.

2. Draw several isoclines of the differential equation $x' = x^2 + t^2$, and from your plots determine, approximately, the graphs of the solution curves.

3. Draw the nullclines for the equation $x' = 1 - x^2$. Graph the isoclines, or the locus of points in the plane where the slope field is equal to -3 and $+3$.

4. Repeat Exercise 2 for the equation $x' = t - x^2$. Find the region in the plane where the slope field is positive and where it is negative.

5. In the right-half tx plane ($t \geq 0$), plot the nullclines of the differential equation $x' = 2x^2(x - 4\sqrt{t})$. Determine the sign of the slope field in the regions separated by the nullclines. Sketch the approximate solution curve passing through the point $(1, 4)$. Why can't your curve cross the $x = 0$ axis?

6. Use software to sketch the slope field for the differential equation $x' = x^2 - t$ on the square $-3 < t < 3$, $-3 < x < 3$.

1.2 Antiderivatives

In calculus we face the problem of finding a function $x = x(t)$ whose derivative is a given function $g(t)$. This is the problem of finding the **antiderivative** $x = x(t)$ of $g(t)$. In the language of this text, we want to solve the *differential equation*

$$x' = g(t), \tag{1.7}$$

for the unknown $x = x(t)$, where $g(t)$ is a given continuous function. For example, if

$$x' = t + 1,$$

then

$$x(t) = \int (t+1)dt = \frac{1}{2}t^2 + t + C,$$

where C is an arbitrary constant of integration. This one-parameter family of curves are integral curves of the differential equation and represent the general solution. In general, we can use the indefinite integral notation and write the solution to (1.7) as

$$x(t) = \int g(t)dt + C. \tag{1.8}$$

It is clear that the antiderivative is unique only up to an additive constant. We can select a unique antiderivative by imposing an initial condition of the form

$$x(t_0) = x_0. \tag{1.9}$$

In other words we want to find the antiderivative that passes through the specific point (t_0, x_0) of the tx plane. This initial condition determines the arbitrary constant C in (1.8). The differential equation (1.8) along with (1.9) is an initial value problem.

Example 1.9

Find $x = x(t)$ that satisfies the initial value problem

$$x' = t^2 - 1, \quad x(1) = 2. \tag{1.10}$$

By direct integration,

$$x(t) = \int (t^2 - 1)dt + C = \frac{1}{3}t^3 - t + C,$$

where C is an arbitrary constant. This is the general solution and it plots as a family of cubic curves in the tx plane, one curve for each value of C. A particular solution to the IVP is found by imposing the initial condition. It selects out a specific value of the constant C, and hence a specific curve. Here, $x(1) = 2$, which gives $\frac{1}{3}(1)^3 - 1 + C = 2$, or $C = \frac{8}{3}$. Therefore, the solution to the initial value problem (1.10) is

$$x(t) = \frac{1}{3}t^3 - t + \frac{8}{3}. \quad \square$$

Example 1.10

For equations of the form $x'' = g(t)$ we can take two successive antiderivatives to find the general solution. Consider the DE

$$x'' = t + 2.$$

Then, integrating once, we get

$$x' = \frac{1}{2}t^2 + 2t + C_1.$$

Integrating again,

$$x = \frac{1}{6}t^3 + t^2 + C_1 t + C_2.$$

Here C_1 and C_2 are two arbitrary constants. For second-order equations we always expect two arbitrary constants, or a two-parameter family of solutions. It takes two initial conditions to determine the arbitrary constants. For example, if $x(0) = 1$ and if $x'(0) = 0$, then $C_1 = 0$ and $C_2 = 1$, and we obtain the particular solution $x(t) = \frac{1}{6}t^3 + t^2 + 1$. $\square$

Consider another example.

Example 1.11

Solve the initial value problem on the interval $t > 0$:

$$x' = e^{-t^2}, \quad x(0) = 2.$$

The general solution of the DE is

$$x(t) = \int e^{-t^2} dt + C.$$

But now we are stymied; there is no formula that gives an expression for the indefinite integral. Therefore, there is no way to use the initial condition to evaluate the constant of integration, or even evaluate the solution $x(t)$ at a particular value of t. In fact, the indefinite integral $\int e^{-t^2} dt$ carries no information, and it is just a notation for the antiderivative. $\square$

To resolve this problem we need a more usable version of the fundamental theorem of calculus, which is a basic result used regularly in differential equations. It provides an expression for the antiderivative of a function in terms of a *definite* integral with a variable upper limit. You should consult your calculus text.

Theorem 1.12

(**Fundamental Theorem of Calculus**) If $g(t)$ is a continuous function, the derivative of an integral with variable upper limit is

$$\frac{d}{dt}\int_a^t g(s)ds = g(t),$$

where the lower limit a is any number. □

A simple illustration of Theorem (1.12) is

$$\frac{d}{dt}\int_2^t \sin\left(\sqrt{1+s^2}\right) ds = \sin\left(\sqrt{1+t^2}\right).$$

The theorem states that the function $G(t) = \int_a^t g(s)ds$ is an antiderivative of g, i.e., a function whose derivative is g. Therefore,

$$G(t) = \int_a^t g(s)ds + C$$

is also an antiderivative for any value of C. This last expression is the *most general form* of the antiderivative of $g(t)$.

Functions defined by integrals are common in the applied sciences and are equally important as functions defined by simple algebraic formulas. To the point, in most calculus texts the natural logarithm $\ln t$ is defined as the inverse of the exponential function, e^t. However, in other texts the natural logarithm is defined by the integral

$$\ln t = \int_1^t \frac{1}{s}\, ds, \quad t > 0.$$

An important perspective is that differential equations often define special functions. For example, the initial value problem

$$x' = \frac{1}{t}, \quad x(1) = 0,$$

can be used to define the natural logarithm function $\ln t$. Many special functions encountered in mathematical physics and engineering are defined by integrals.

Example 1.13

We return to the initial value problem in Example 1.11:

$$x' = e^{-t^2}, \quad x(0) = 2.$$

From the fundamental theorem of calculus we write the antiderivative as

$$x(t) = \int_0^t e^{-s^2} ds + C.$$

The common strategy is to take the lower limit of integration to be the initial value of t, here zero. Then $x(0) = 0 + C = 2$, or $C = 2$. We obtain the solution to the initial value problem in the form of a definite integral with a variable upper limit of integration,

$$x(t) = \int_0^t e^{-s^2} ds + 2.$$

Calculators and computer algebra systems plot such functions easily. □

Some functions defined by integrals are often given a name, particularly if they occur frequently. For example, we can define the special function "erf" (called the **error function**) by

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds. \qquad \text{(error function)}$$

The factor $2/\sqrt{\pi}$ in front of the integral normalizes the function so that $\text{erf}(+\infty) = 1$. The erf function $\text{erf}(t)$ gives the area under the bell-shaped curve $(2/\sqrt{\pi})\exp(-s^2)$ from 0 to t. The erf function, which is plotted in Figure 1.6, is an important function in probability and statistics, and in diffusion processes. Its values are tabulated in computer algebra systems and mathematical handbooks.

EXERCISES

1. Find the general solution to the differential equation

   $$x' = t\cos(t^2).$$

   Find the particular solution satisfying the initial condition $x(0) = 1$, and plot the solution on the interval $-5 \le t \le 5$.

2. Solve the initial value problem $x' = (t+1)/\sqrt{t}$, $x(1) = 4$.

3. Find a function $x(t)$ that satisfies the initial value problem $x'' = -3\sqrt{t}$, $x(1) = 1$, $x'(1) = 2$.

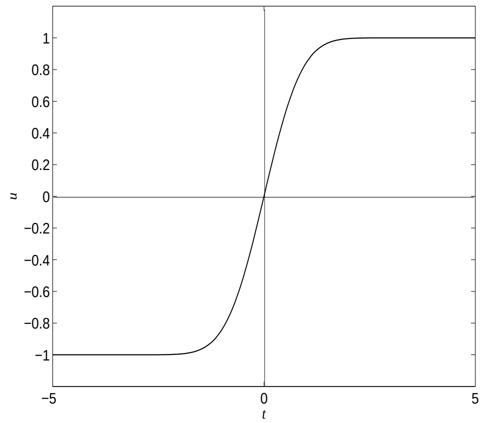

Figure 1.6 Plot of the erf function on the interval [-5,5] using the MATLAB commands: t=-5:0.01:5; u=erf(t); plot(t,u).

4. Find all functions that solve the differential equations:

   (a) $x' = te^{-2t}$. (b) $x' = 1/(t \ln t)$. (c) $\sqrt{t}\, x' = \cos \sqrt{t}$.

5. (*Mechanics*) A car of mass m is moving at speed V when it has to brake. The brakes apply a constant force F until the car comes to rest. How long does it take the car to stop? How far does the car go before stopping? Now, with specific data, compare the time and distance it takes to stop if you are going 30 mph versus 35 mph. Take $m = 1000$ kg and $F = 6500$ N. Write a short paragraph on recommended speed limits in residential areas.

6. Find the solution to the initial value problem $x' = e^{-t}/\sqrt{t}$, $x(1) = 0$, in terms of an integral with a variable upper limit. Plot the solution on the interval $[1, 4]$ using a computer algebra system.

7. Solve the initial value problem for $x = x(t)$:

$$\frac{d}{dt}\left(t\frac{dx}{dt}\right) = 1, \quad x(1) = 1, \;\; x'(1) = 2.$$

8. (*Physics*) A ball of mass m is tossed upward from a building at height h with initial velocity v. The only force acting on the ball is the force of gravity, $F = -mg$, where g is the acceleration due to gravity. Let $x = x(t)$ denote the height above ground level at time t. By Newton's second law the initial value problem is

$$mx'' = -mg, \quad x(0) = h, \;\; x'(0) = v.$$

Solve the IVP and show that the height is given by the familiar formula

$$x(t) = -\frac{1}{2}gt^2 + vt + h.$$

9. The differential equation $x' = 3x + e^{-t}$ can be converted into an antiderivative problem for a new dependent variable $y = y(t)$ using the substitution $x(t) = y(t)e^{3t}$. Find the differential equation for y, solve it, and then determine the general solution $x = x(t)$ of the original equation.

10. Use the chain rule and the fundamental theorem of calculus to compute the derivative of the following functions:

    (a) $f(t) = \text{erf}(\sin t)$. (b) $f(t) = t\int_1^t e^{-\sqrt{s}}ds$.

11. A function is defined by the expression

$$y(t) = e^{-t^2}\int_0^t e^{s^2}ds.$$

    Find a differential equation for $y = y(t)$.

12. An enemy cannon at horizontal distance L from a fort can fire a cannon ball from the top of a hill at height H above the ground level with a muzzle velocity v. How high should the wall of the fort be to guarantee that a cannon ball does not go over the wall? Observe that the enemy can adjust the angle θ of its shot. Hint: Ignoring air resistance, the governing equations follow from resolving Newton's second law for the horizontal and vertical components of the force: $mx'' = 0$ and $my'' = -mg$.

13. An **integral equation** is an equation where the unknown $x = x(t)$ appears under an integral sign. Such equations arise in many applications. An example is the equation

$$x(t) = e^{-2t} + \int_0^t sx(s)ds.$$

    Transform this equation into an initial value problem for $x(t)$. Hint: Differentiate.

14. Transform the integral equation

$$x(t) + \int_0^t e^{-a(t-s)}x(s)ds = b; \quad a,\ b \quad \text{constants},$$

    into an initial value problem and find the solution $x = x(t)$.

15. Show how the initial value problem $x' = f(t,x)$, $x(0) = x_0$, can be transformed into the integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s))ds.$$

Transform the initial value problem

$$x' = 5tx^2 + 1, \quad x(1) = 0$$

into an integral equation.

16. Using your calculator, plot the function

$$x(t) = \int_0^t e^{-s^2} ds + 2, \quad 0 \le t \le 4.$$

17. (*Leibniz rule*) Define the function $I(t,b) = \int_a^b f(t,s)ds$ of two variables. Then

$$\frac{\partial I}{\partial t} = \int_a^b f_t(t,s)ds, \qquad \frac{\partial I}{\partial b} = f(t,b).$$

Use the chain rule to show Leibniz rule:

$$\frac{d}{dt} I(t, b(t)) = \int_a^{b(t)} f_t(t,s)ds + f(t, b(t))b'(t).$$

1.3 Separable Equations

1.3.1 Separation of Variables

A differential equation having the form

$$x' = f(x)g(t), \tag{1.11}$$

where the right side is a product of a function of x and a function of t, is called a **separable equation**.

The procedure to solve (1.11) is simple. To do so we usually write (1.11) in the notation

$$\frac{dx}{dt} = f(x)g(t).$$

We divide by $f(x)$ to obtain

$$\frac{1}{f(x)} \frac{dx}{dt} = g(t).$$

Integrating with respect to t (remember, x is a function of t) we get

$$\int \frac{1}{f(x)}\frac{dx}{dt}\,dt = \int g(t)\,dt.$$

But $x = x(t)$, so $dx = \frac{dx}{dt}\,dt$. Then

$$\int \frac{1}{f(x)}dx = \int g(t)dt + C, \tag{1.12}$$

where C is an arbitrary constant. After calculating the two integrals, if possible, we obtain a one-parameter family of curves $\phi(t,x) = C$ in x and t, called the **integral curves** of (1.11); we get one curve for each value of C. The integral curves define **implicit solutions** of the equation. To find **explicit solutions** we must solve for $x = x(t)$ in terms of t and C. If there is an initial condition, it determines the value of C.

Example 1.14

Solve the equation

$$\frac{dx}{dt} = \frac{t}{x}.$$

Separating the variables we get

$$x\frac{dx}{dt} = t.$$

Integrating,

$$\int x\frac{dx}{dt}\,dt = \int t\,dt,$$

or, since $dx = \frac{dx}{dt}\,dt$,

$$\int x\,dx = \int t\,dt,$$

or

$$\frac{1}{2}x^2 = \frac{1}{2}t^2 + C.$$

This one-parameter family of integral curves, which are hyperbolas in the tx plane, define the solutions $x = x(t)$ implicitly. To find explicit solutions we must solve for x. Multiplying by 2 and taking the square root of both sides gives

$$x(t) = \pm\sqrt{t^2 + C},$$

where we have replaced $2C$ by C. (2 times an arbitrary constant C is just another arbitrary constant, which we again call C.)

If we are given an initial condition, say

$$x(2) = 1,$$

then $x(2) = \pm\sqrt{2^2 + C} = 1$, then we are forced to take the *positive* square root and obtain $C = -3$. Therefore, the solution to the IVP

$$\frac{dx}{dt} = \frac{t}{x}, \quad x(2) = 1$$

is

$$x(t) = \sqrt{t^2 - 3}.$$

The solution to the IVP is defined only when $t > \sqrt{3}$, the interval of existence. □

Remark 1.15

(**Recipe**) The method of separation of variables for the equation

$$\frac{dx}{dt} = f(x)g(t)$$

just results in writing down

$$\frac{1}{f(x)}\ dx = g(t)\ dt,$$

where the x terms are placed on the left and the t terms on the right, including the two differentials dx and dt. Then we integrate to get

$$\int \frac{1}{f(x)} dx = \int g(t) dt + C.$$

This is the recipe used to solve problems. We usually dispense with integrating both sides with respect to t and then changing variables. We go directly to the last equation. □

Example 1.16

(**Growth and Decay**) Consider the differential equation

$$\frac{dx}{dt} = rx, \tag{1.13}$$

where r is a given constant. If $r < 0$ this is the decay equation; if $r > 0$ then it models exponential growth. This equation is separable. We write

$$\frac{1}{x} dx = r dt.$$

Integrating gives

$$\int \frac{1}{x} dx = r \int dt,$$

or

$$\ln|x| = rt + C \quad \text{or} \quad |x| = e^{rt+C} = e^C e^{rt}.$$

This means $x = \pm e^C e^{rt}$. Therefore, the general solution of the growth–decay equation can be written compactly as

$$x(t) = Ce^{rt},$$

where $\pm e^C$ has been replaced by C. This is the same solution we obtained earlier in a different way. □

Remark 1.17

Sometimes it is convenient in an initial value problem to integrate both sides of the equation using definite integrals with variable upper limits. For example, consider the IVP

$$\frac{dx}{dt} = g(t)f(x), \quad x(t_0) = x_0,$$

Then we write

$$\frac{1}{f(x)}dx = g(t)dt,$$

and then integrate both sides as follows:

$$\int_{x_0}^{x} \frac{1}{f(y)}dy = \int_{t_0}^{t} g(s)ds,$$

where we have introduced two dummy variables, one in each integrand. Then we calculate the integrals as usual and obtain, after simplification, the solution $x = x(t)$ to the IVP. No constants of integration are introduced. □

Example 1.18

Solve the initial value problem for $t > 1$:

$$\frac{dx}{dt} = \frac{2\sqrt{x}e^{-t}}{t}, \quad x(1) = 4.$$

The equation is separable so we separate variables and integrate:

$$\frac{1}{2}\int_4^t \frac{1}{\sqrt{y}}\, dy = \int_1^t \frac{e^{-s}}{s}\, ds.$$

We can integrate the left side exactly, but the integral on the right cannot be resolved in closed form. Integrating the left side then gives

$$\sqrt{x} - \sqrt{4} = \int_1^t \frac{e^{-s}}{s}\, ds.$$

Solving for x gives

$$x(t) = \left(\int_1^t \frac{e^{-s}}{s}\, ds + 2 \right)^2.$$

This solution is valid on $1 \leq t < \infty$. In spite of the apparent complicated form of the solution, which contains an integral, it is not difficult to plot using a calculator. □

EXERCISES

1. Use the method of separation of variables to find the general solution to the following differential equations.

   (a) $x' = \sqrt{x}$.

   (b) $x' = e^{-2x}$.

   (c) $y' = 1 + y^2$.

   (d) $u' = \dfrac{1}{5 - 2u}$.

   (e) $x' = au + b, \quad a, b > 0$.

   (f) $Q' = \dfrac{Q}{4 + Q^2}$.

   (g) $x' = e^{x^2}$.

   (h) $y' = r(a - y)$.

2. Solve $y' = r(a - y)$, where r and a are constants.

3. In Exercises 1(a)–(b) find the solution to the resulting IVP when $x(0) = 1$.

4. Find the general solution:

   (a) $x' = \dfrac{2x}{t + 1}$.

   (b) $\theta' = t\sqrt{t^2 + 1}\, \sec\theta$.

   (c) $(2u + 1)u' - (t + 1) = 0$,

   (d) $R' = (t + 1)(R^2 + 1)$.

   (e) $y' + y + \dfrac{1}{y} = 0$.

   (f) $(t + 1)x' + x^2 = 0$,

5. Solve the initial value problem and find the interval of existence.

$$\frac{dy}{dt} = \frac{1}{2y + 1}, \quad y(0) = 1.$$

6. Find the interval of existence for the initial value problem

$$\frac{dx}{dt} = (4t - x)^2, \quad x(0) = 1.$$

   Hint: Change the dependent variable to $y = y(t)$ where $y = 4t - x$.

7. Determine the maximum interval of existence of the solution $x = x(t)$ to

$$x' = 2tx^2, \quad x(0) = 1.$$

8. Find the solution to the initial value problem

$$x' = t^2 e^{-x}, \quad x(0) = \ln 2,$$

and determine the interval of existence.

9. Solve $x' = x(4+x)$ subject to the initial condition $x(0) = 1$. Hint: It is helpful to use a partial fractions expansion

$$\frac{1}{x(4+x)} = \frac{a}{x} + \frac{b}{4+x},$$

where a and b are to be determined.

10. Solve the following initial value problems.:

    a) $\frac{dx}{dt} = e^{t+x}, \quad x(0) = 0.$

    b) $\frac{dT}{dt} = 2at(T^2 - a^2), \quad T(0) = 0.$

    c) $\frac{dy}{dt} = t^2 \tan y, \quad y(0) = 0.$

11. Find the general solution in implicit form to the equation

$$x' = \frac{(4+2t)x}{\ln x}.$$

Find the solution when $x(0) = e$ and plot the result. What is the interval of existence?

12. Solve the initial value problem

$$y' = \frac{2ty^2}{1+t^2}, \quad y(0) = y_0$$

and find the interval of existence when $y_0 < 0$, when $y_0 > 0$, and when $y_0 = 0$.

13. The integral curves of the differential equation

$$\frac{dx}{dt} = \frac{t^2}{1-x^2}$$

are $-t^3 + 3x - x^3 = C$. **(a)** Using implicit differentiation, verify that the integral curves represent solutions. **(b)** Find the curve passing through $x = 1$ at $t = 1$, and plot it in the tx plane. Hint: Plot t versus x. (c) Does the initial value problem with initial condition $x(1) = 1$ have only one solution?

14. A differential equation has integral curves

$$x^2 + 2t^2 = C.$$

(a) Sketch several of these curves in the tx plane. **(b)** Find the differential equation. **(c)** Find the explicit solution satisfying the condition $t = 1$, $x = 4$ and determine its interval of existence.

15. Find the general solution of the DE

$$x' = 6t(x-1)^{2/3}.$$

Clearly, $x(t) = 1$ is a constant solution. Solve the DE and show that there is no value of the arbitrary constant that gives the solution $x(t) = 1$. (A solution to a DE that cannot be obtained from the general solution by fixing a value of the arbitrary constant is called a **singular solution.**)

16. (*Mechanics*) We modeled the velocity of an object falling in a fluid by the equation $mv' = mg - av^2$. Use separation of variables and partial fractions to show that the general solution is

$$v(t) = \frac{rm}{a}\left(\frac{e^{rt} - Ce^{-rt}}{e^{rt} + Ce^{-rt}}\right), \quad r^2 = \frac{ag}{m}.$$

If $v(0) = V$, determine the constant C and then find the limiting, or terminal, velocity.

17. Find the general solution to the logistic equation $x' = rx(1 - x/K)$ using separation of variables. Hint: use the partial fractions decomposition

$$\frac{1}{x(K-x)} = \frac{1/K}{x} + \frac{1/K}{K-x}.$$

18. (*Population Growth*) In this exercise derive the logistic model in an alternate way. Suppose the per capita growth rate of a population $x = x(t)$ is the birth rate minus the death rate, or $r - c_i x$, where r is the birth rate and c_i is the coefficient of intra-specific (internal, within the population) competition. As the population increases there is greater competition for the existing resources, which decreases the growth rate and limits growth. Define c_i by

$$c_i = \frac{\text{demand for resources}}{\text{total resources}} = \frac{D}{H}.$$

The dimensions of D are resources/time per animal, and H is given in resources. Derive the logistic law, and show that the carrying capacity is $K = rH/D$, given in animals.

19. (*Tumor growth*) One model of tumor growth is the Gompertz equation

$$\frac{dR}{dt} = -aR\ln(R/k),$$

where $R = R(t)$ is the tumor radius, and a and k are positive constants. Solve the Gompertz equation for $R(t)$?

20. (*Allometry*) Allometric growth describes temporal relationships between sizes of different parts of organisms as they grow (e.g., the leaf area and the stem diameter of a plant). We say two sizes x and y are *allometrically* related if their relative growth rates are proportional, or

$$\frac{x'}{x} = a\frac{y'}{y}, \quad a > 0.$$

Show that if x and y are allometrically related, then $x = Ky^a$, for some constant K.

21. (*Homogeneous Equations*) A differential equation of the form

$$\frac{dx}{dt} = F\left(\frac{x}{t}\right),$$

where the right side depends only on the ratio of x and t, is called **homogeneous**. Show that a homogeneous equation can be transformed into a separable equation by changing the dependent variable from x to y via $x(t) = ty(t)$. Use this method of substitution to solve the equation

$$\frac{dx}{dt} = \frac{4t^2 + 3x^2}{2tx}.$$

22. Show that the differential equation

$$\frac{dx}{dt} = F(at + bx + c)$$

can be transformed into a separable equation by making a transformation $y = at + bx + c$ to a new dependent variable $y = y(t)$. Solve the following equations:

$$\text{(a)} \quad \frac{dx}{dt} = (t + x)^2. \qquad \text{(b)} \quad \frac{dx}{dt} = \sqrt{2t + x + 3}.$$

23. Solve the initial value problem for $x = x(t)$:

$$\frac{d}{dt}\left(xe^{2t}\right) = e^{-t}, \quad x(0) = 3.$$

Hint: Integrate both sides.

24. Find the general solution $x = x(t)$ of the DE

$$\frac{1}{t}\frac{d}{dt}\left(tx'(t)\right) = -2.$$

25. (*Epidemiology*) A population of u_0 individuals all have HIV, but none has the symptoms of AIDS. Let $u(t)$ denote the number that does not have AIDS at time $t > 0$. If $r(t)$ is the per capita rate of individuals showing AIDS symptoms (the conversion rate from HIV to AIDS), then $u'/u = -r(t)$. In the simplest case we can take r to be a linear function of time, or $r(t) = at$. Find $u(t)$ and sketch the solution when $a = 0.2$ and $u_0 = 100$. At what time is the rate of conversion maximum?

26. Find the general solution of

$$\frac{dy}{dt} = \frac{y^2 + 2ty}{t^2}.$$

27. In very cold weather the thickness of ice on a pond increases at a rate inversely proportional to its thickness. If the ice initially is 0.05 inches thick and 4 hours later it is 0.075 inches thick, how thick will it be in 10 hours?

28. Write the solution to the initial value problem

$$\frac{dy}{dt} = -y^2 e^{-t^2}, \quad y(0) = \frac{1}{2}$$

in terms of the erf function.

29. (*Demography*) Let N_0 be the number of individuals in a cohort at time $t = 0$ and $N = N(t)$ be the number of those individuals that are still alive at time t. If m is the constant per capita mortality rate, then $N'/N = -m$, which gives $N(t) = N_0 e^{-mt}$. The *survivorship function* is defined by $S(t) = N(t)/N_0$, and $S(t)$ therefore gives the probability of an individual living to age t. In the case of a constant per capita mortality the survivorship curve is a decaying exponential function $S(t) = e^{-mt}$.

    a) What fraction of the cohort die before age t? Calculate the fraction that die between age a and age b.

    b) If the per capita death rate depends on time (or age), or $m = m(t)$, find a formula for the survivorship function (your answer will contain an integral).

    c) What do you think the human survivorship curve $S(t)$ might look like? What about an insect's survivorship curve?

1.3.2 Heat Transfer

A small object of uniform temperature T_0 is placed in an environment oven of constant temperature T_e. (E.g., a chemical sample, or even a potato, is placed in an oven.) Over time the object heats up and eventually its temperature is that of the environment, T_e. We want a model that governs the temperature $T(t)$ of the object at any time t. **Newton's law of cooling** (or, heating), states that the rate of change of the temperature of the object is proportional to the difference between the temperature of the object and the environmental temperature. That is,

$$T' = -h(T - T_e). \tag{1.14}$$

The positive proportionality constant h is the *heat loss coefficient* and it measures how fast an object releases or absorbs heat. There is a fundamental assumption that the heat is instantly and uniformly distributed throughout the object and there are no temperature gradients, or spatial variations, in the body itself. From the DE we observe that $T = T(t) = T_e$ is a constant solution. Because it is not changing, it is called an *equilibrium solution.* If $T > T_e$ then $T' < 0$, and the temperature decreases; if $T < T_e$ then $T' > 0$, and the temperature increases. We can easily find a formula for the temperature $T(t)$

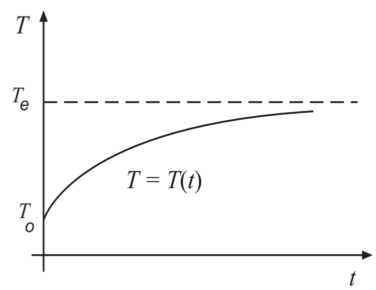

Figure 1.7 Temperature history in Newton's law of cooling showing how the temperature $T(t)$ approaches the equilibrium temperature when $T_0 < T_e$.

satisfying (1.14) using separation of variables. Write the equation as

$$\frac{dT}{T - T_e} = -hdt.$$

Integrating gives

$$\ln \mid T - T_e \mid = -ht + C.$$

Exponentiating both sides gives

$$\mid T - T_e \mid = e^{-ht} e^C,$$

or, renaming the arbitrary constant,

$$T - T_e = Ce^{-ht}$$

Hence,

$$T(t) = T_e + Ce^{-ht},$$

which is the general solution of (1.14) containing an arbitrary constant C. When we impose an initial condition $T(0) = T_0$, then we find $C = T_0 - T_e$, giving the particular solution to the initial value problem:

$$T(t) = T_e + (T_0 - T_e)e^{-ht}.$$

Immediately we observe that $T(t) \to T_e$ as $t \to \infty$. A plot of the solution showing how an object heats up is illustrated in Figure 1.7.

EXERCISES

1. A small turkey at room temperature 70°F is placed into an oven at 350°F. If $h = 0.42$ per hour is the heat loss coefficient for turkey meat, how long should you cook the turkey so that it is uniformly 200°F? Comment on the validity of the assumptions being made in this model?

2. A pan of water at 46°C was put into a refrigerator. Ten minutes later the water was 39°C, and ten minutes after that it was 33°C. Estimate the temperature inside the refrigerator.

3. (*Forensics*) The body of a murder victim was discovered at 11:00 A.M. The medical examiner arrived at 11:30 A.M. and found the temperature of the body was 94.6°F. The temperature of the room was 70°F. One hour later, in the same room, he took the body temperature again and found that it was 93.4°F. Estimate the time of death. What assumptions are being made?

4. (*Home heating*) Suppose the temperature inside your winter home is 68°F at 1:00 P.M. and your furnace then fails. If the outside temperature is 10°F and you notice that by 10:00 P.M. the inside temperature is 57°F, what is the temperature in your home the next morning at 6:00 A.M.?

1.3.3 Chemical Reactors

A *continuously stirred tank reactor* (also called a chemostat, or compartment) is a basic unit of many physical, chemical, and biological processes. It is a well-defined geometric volume where substances enter, react, and then discharged.

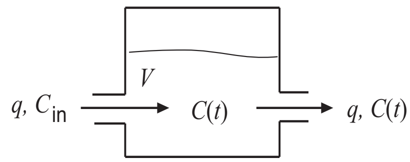

Figure 1.8 A chemostat, or continuously stirred tank reactor.

A chemostat could be an organ in our body, a polluted lake, an industrial chemical reactor, or even an ecosystem. See Figure 1.8.

We illustrate a reactor model with a specific example. Consider an industrial pond with constant volume V cubic meters. Suppose that polluted water containing a toxic chemical of concentration $C_{\rm in}$ grams per cubic meter is dumped into the pond at a constant volumetric flow rate of q cubic meters per day. At the same time the continuously mixed solution in the pond is drained off at the same flow rate q. If the pond is initially at concentration C_0, what is the concentration $C(t)$ of the chemical in the pond at any time t?

The key idea in all chemical mixture problems is to obtain a model by conserving mass: the rate of change of mass in the pond must equal the rate mass flows in minus the rate mass flows out. The total mass in the pond at any time is VC, and the mass flow rate is the volumetric flow rate times the mass concentration; thus mass balance dictates

$$(VC)' = qC_{\rm in} - qC.$$

Hence, the initial value problem for the chemical concentration is

$$VC' = qC_{\rm in} - qC, \quad C(0) = C_{0}, \tag{1.15}$$

where C_0 is the initial concentration in the tank. This initial value problem can be solved by the separation of variables method.

A similar reactor model holds when the volumetric flow rates in and out are different, which gives a changing volume $V(t)$. Letting $q_{\rm in}$ and $q_{\rm out}$ denote those flow rates, respectively, we have

$$(V(t)C)' = q_{\rm in}C_{\rm in} - q_{\rm out}C,$$

where $V(t) = V_0 + (q_{\rm in} - q_{\rm out})t$, and where V_0 is the initial volume. Methods developed in Section 2.1 show how to handle this equation.

Now suppose we add degradation of the chemical while it is in the pond, assuming that it degrades to inert products at a rate proportional to the amount present. We represent this decay rate as kC gm per cubic meter per day, where k is the rate constant in units of time^{-1}. Then the model equation becomes

$$VC' = qC_{\rm in} - qC - kVC.$$

Notice that we include a factor V in the last term to make the model dimensionally correct.

Also, the chemical can be consumed or created in the reactor by a chemical reaction. The law of mass action from chemistry dictates the rate of the reaction. The exercises present some examples.

EXERCISES

1. Solve the initial value problem (1.15) and obtain a formula for the concentration in the reactor at time t.

2. (*Pollution*) An industrial pond having volume 100 m^3 is full of pure water. Contaminated water containing a toxic chemical of concentration 0.0002 kg per m^3 is then is pumped into the pond with a volumetric flow rate of 0.5 m^3 per minute. The contents are well-mixed and pumped out at the same flow rate. Write down an initial value problem for the contaminant concentration $C(t)$ in the pond at any time t. Determine the equilibrium concentration and its stability. Find a formula for the concentration $C(t)$.

3. In the preceding problem, change the flow rate out of the pond to 0.6 m^3 per minute. How long will it take the pond to empty? Write down, but do not solve, the revised initial value problem.

4. A vat of volume 1000 gallons initially contains 5 lbs of salt. For $t > 0$ pure water is pumped into the vat at the rate of 2 gallons per minute; the perfectly stirred mixture is pumped out at the same flow rate. Derive a formula for the concentration of salt in the tank at any time t. Sketch a graph of the concentration versus time.

5. A vat of volume 1000 gallons initially contains 5 lbs of salt. For $t > 0$ a salt brine of concentration 0.1 lbs per gallon is pumped into the tank at the rate of 2 gallons per minute; the perfectly stirred mixture is pumped out at the same flow rate. Derive a formula for the concentration of salt in the tank at any time t. Sketch a graph of the concentration versus time.

6. Consider a chemostat of constant volume where a chemical C is pumped into the reactor at constant concentration and constant flow rate. While in the reactor it reacts according to $C + C \to$ products. From the law of mass action the rate of the reaction is $r = kC^2$, where k is the rate constant. If the concentration of C in the reactor is given by $C(t)$, then mass balance leads the governing equation $(VC)' = qC_{\text{in}} - qC - kVC^2$. Find the constant solutions, or equilibrium states, of this equation. What are the units of k?

7. (*Enzyme kinetics*) Work Exercise 6 if the rate of an enzyme reaction is

given by *Michaelis–Menten kinetics*

$$r = \frac{aC}{b + C},$$

where a and b are positive constants.

8. (*Batch reactor*) A batch reactor is a reactor of volume V where there are no in and out flow rates. Reactants are loaded instantaneously and then allowed to react over a time T, called the residence time. Then the contents are expelled instantaneously. Fermentation reactors and even sacular stomachs of some animals can be modeled as batch reactors. If a chemical is loaded in a batch reactor and it degrades with rate $r(C) = kC$, given in mass per unit time, per unit volume, what is the residence time required for 90 percent of the chemical to degrade?

9. (*Reaction kinetics*) Consider the chemical reaction $\mathbf{A} + \mathbf{B} \xrightarrow{k} \mathbf{C}$, where one molecule of **A** reacts with one molecule of **B** to produce one molecule of **C**, and the rate of the reaction is k, the rate constant. By the *law of mass action* in chemistry, the reaction rate is $r = kab$, where a and b represent the time-dependent concentrations of the reactants **A** and **B**. Thus, the rates of change of the reactants and product are governed by the three equations

$$a' = -kab, \quad b' = -kab, \quad c' = kab.$$

Initially, $a(0) = a_0$, $b(0) = b_0$, and $c(0) = 0$, with $a_0 > b_0$. Show that $a - b =$ constant $= a_0 - b_0$, and find a single, first-order differential equation that involves only the concentration $a = a(t)$. What is the limiting concentration $\lim_{t\to\infty} a(t)$? What are limiting concentrations of **B** and **C**?

10. (*Digestion) Digestion* in the stomach (gut) in some simple organisms can be modeled as a chemical reactor of volume V, where food enters and is broken down into nutrient products, which are then absorbed across the gut lining; the food–product mixture in the stomach is perfectly stirred and exits at the same rate as it entered. Let S_0 be the concentration of a substrate (food) consumed at rate q (volume per time). In the gut the rate of substrate breakdown into the nutrient product, $S \to P$, is given by kVS, where k is the rate constant and $S = S(t)$ is the substrate concentration. The nutrient product, of concentration $P = P(t)$, is then absorbed across the gut boundary at a rate aVP, where a is the absorption constant. At all times the contents are thoroughly stirred and leave the gut at the flow rate q.

   a) Argue that the model equations are

$$VS' = qS_0 - qS - kVS, \quad VP' = kVS - aVP - qP.$$

b) Suppose the organism eats continuously, in a steady-state mode, where the concentrations become constant. Find the constant solutions, or equilibrium concentrations S_e and P_e.

c) Some ecologists believe that animals regulate their consumption rate in order to maximize the absorption rate of nutrients. Show that the maximum nutrient concentration P_e occurs when the consumption rate is $q = \sqrt{ak}V$.

d) Show that the maximum absorption rate is therefore

$$\frac{akS_0V}{(\sqrt{a}+\sqrt{k})^2}.$$

1.4 Linear Equations

1.4.1 Integrating Factors

A differential equation of the form

$$x' + p(t)x = q(t). \tag{1.16}$$

is called a **first-order linear equation**. If a first-order equation cannot be put into the form (1.16), the equation is called **nonlinear**. Equation (1.16) is called the **normal form** of a first-order linear equation. The procedure we present to solve a linear equation requires that it be in normal form, with the coefficient of x' equal to 1.

Example 1.19

The equation

$$(\sin t)x' = 3tx + \sqrt{t}$$

is linear, but not in normal form. But we can divide by $\sin t$ and write it as

$$x' - \frac{3t}{\sin t}x = \frac{\sqrt{t}}{\sin t},$$

with $p(t) = 3t/\sin t$ and $q(t) = \sqrt{t}/\sin t$. It is now in normal form. The equation

$$x' + (\cos t)xx' = 2$$

is nonlinear. Clearly the equation $t^2x' - 4(t+1) = 0$ is linear and the normal form is

$$x' = \frac{4(t+1)}{t^2}.$$

Here, $p(t)$ is zero. □

For this discussion we assume the prescribed functions p and q in (1.16) are continuous. These equations occur frequently in applications. If $q(t) = 0$, then the equation (1.16) is called **homogeneous**; the homogeneous equation is

$$x' + p(t)x = 0.$$

Observe that the homogeneous equation is separable, and its solution is easily found to be

$$x(t) = Ce^{-\int p(t)dt}, \quad C \text{ an arbitrary constant.}$$

In this context equation (1.16) is called **nonhomogeneous**.

Because of its role in applications, the function $q(t)$ in (1.16) is called the **source term** or **forcing term**.

The left side of a linear equation in normal form, (1.16), has a very nice property: the left side of the equation can be multiplied by a function $\mu = \mu(t)$ that transforms it into a total derivative. In other words, there is a function $\mu = \mu(t)$ such that

$$\mu(t)\left(x' + p(t)x\right) = \left(\mu(t)x\right)'.$$

The function $\mu(t)$ is called an **integrating factor** and it is given by

$$\mu(t) = e^{\int p(t)\,dt} = e^{P(t)}, \quad \text{where} \quad P(t) = \int p(t)\,dt;$$

$P(t)$ is the antiderivative of $p(t)$. By the product rule and the chain rule it is easy to check that it works:

$$\begin{aligned}(\mu(t)x)' &= \left(e^{P(t)}x\right)' \\ &= e^{P(t)}x' + e^{P(t)}p(t)x \qquad (\text{because } P'(t) = p(t)) \\ &= \mu(t)x' + \mu(t)p(t)x = \mu(t)(x' + p(t)x),\end{aligned}$$

which is our claim.

Procedure for solving a linear equation

Step 1. Multiply both sides of the normal form of the equation

$$x' + p(t)x = q(t)$$

by the integrating factor

$$\mu(t) = e^{\int p(t)\,dt} = e^{P(t)}.$$

Step 2. Obtain

$$\left(e^{P(t)}x\right)' = e^{P(t)}q(t).$$

Step 3. Integrate (take the antiderivative) both sides to obtain

$$e^{P(t)}x(t) = \int e^{P(t)}p(t)dt + C.$$

Step 4. Multiply by $e^{-P(t)}$ to obtain the general solution

$$x(t) = e^{-P(t)} \int e^{P(t)}p(t)\, dt + Ce^{-P(t)}. \tag{1.17}$$

Remark 1.20

If any of the integrals required in the procedure cannot be computed in closed form, then they should be written as a definite integral with variable upper limit. For example, the integrating factor

$$\mu(t) = e^{\int p(t)\, dt},$$

can always be written

$$\mu(t) = e^{\int_a^t p(s)\, ds},$$

where we usually take the lower limit a to be the time corresponding to an initial condition. □

Example 1.21

Solve the equation

$$x' + \frac{1}{t}x = 1.$$

Here $p(t) = 1/t$ and $q(t) = 1$. The integrating factor is

$$\mu(t) = e^{\int (1/t)\, dt} = e^{\ln t} = t.$$

Multiplying both sides of the DE by t gives

$$tx' + x = t, \quad \text{or} \quad (tx)' = t.$$

Integrate both sides:

$$tx = \frac{1}{2}t^2 + C.$$

Therefore

$$x(t) = \frac{1}{2}t + \frac{C}{t},$$

which is the general solution. The arbitrary constant C is determined from an initial condition. For example, if the initial condition

$$x(1) = 3,$$

is imposed, then

$$x(1) = \frac{1}{2}(1) + \frac{C}{(1)} = 3, \qquad \text{or} \qquad C = \frac{5}{2}.$$

Therefore the solution to the initial value problem is

$$x(t) = \frac{1}{2}t + \frac{5}{2t}. \quad \square$$

Here is an example involving a more difficult integration.

Example 1.22

Consider the differential equation

$$x' + 2x = \sin t.$$

We multiply the DE by the integrating factor

$$\mu(t) = e^{\int 2dt} = e^{2t}$$

to get

$$\left(xe^{2t}\right)' = e^{2t}\sin t.$$

Integrating both sides gives,

$$xe^{2t} = \int e^{2t}\sin t\,dt + C,$$

or

$$x(t) = e^{-2t}\int e^{2t}\sin t\,dt + Ce^{-2t}.$$

The integral on the right side can be calculated using integration by parts (or consulting an integral table). In any case we obtain the general solution

$$\begin{aligned} x(t) &= e^{-2t}\left[e^{2t}\left(\frac{2}{5}\sin t - \frac{1}{5}\cos t\right)\right] + Ce^{-2t} \\ &= \frac{2}{5}\sin t - \frac{1}{5}\cos t + Ce^{-2t}. \quad \square \end{aligned}$$

The solution (1.17) to the first order linear equation has a general structure that it shares with other linear nonhomogeneous equations of higher order. The result is easily obtained by comparing it to (1.17).

Theorem 1.23

(**Structure Theorem**) Consider the first-order linear equation

$$x' + p(t)x = q(t).$$

The general solution $x(t)$ is the sum of the general solution to the homogeneous equation plus any solution to the nonhomogeneous equation. That is, it is the sum of the homogeneous solution and a particular solution:

$$x(t) = x_h(t) + x_p(t),$$

where

$$x_h(t) = Ce^{-P(t)}, \quad x_p(t) = e^{-P(t)} \int q(t)e^{P(t)} dt. \quad \square$$

Therefore, the solution consists of two parts: a so-called *transient solution*

$$x_h(t) = Ce^{-P(t)}$$

involving the initial condition, which determines C, and a so-called *steady-state solution*

$$x_p(t) = e^{-P(t)} \int q(t)e^{P(t)} \, dt$$

involving the nonhomogeneous term $q(t)$. In mathematical jargon, $x_h(t)$ is called the *homogeneous solution* (or the *complementary solution* in some texts) because it satisfies the homogeneous equation; $x_p(t)$ is called a *particular solution* because it is a specific solution to the nonhomogeneous equation.

Example 1.24

Consider the DE

$$x' - 3x = e^{-t}.$$

The integrating factor is $\mu(t) = \exp(\int -3dt) = e^{-3t}$. Multiplying through by the integrating factor, the DE becomes or

$$\left(xe^{-3t}\right)' = e^{-4t}.$$

Integrating both sides gives

$$xe^{-3t} = C - \frac{1}{4}e^{-4t},$$

or

$$x(t) = Ce^{3t} - \frac{1}{4}e^{-t},$$

which is the general solution. It is the sum of the homogeneous solution and a particular solution. The solution to the homogeneous equation $x' - 3x = 0$ is

$$x_h(t) = Ce^{3t},$$

which is a growing exponential solution. The particular solution is

$$x_p(t) = -\frac{1}{4}e^{-t},$$

which involves the nonhomogeneous term; it decays away as $t \to +\infty$. (The reader should check that this is a solution to the nonhomogeneous equation $x' - 3x = e^{-t}$.) □

Example 1.25

Observe that the differential equation $x' = ax + b$ can be solved by three different methods: separation of variables, integrating factors, or making the substitution $y = ax + b$. See Exercise 9. □

EXERCISES

1. Classify the first-order equations as linear or nonlinear.

   a) $x' = 2t^3x - 6$.
   b) $(\cos t)x' - 2x \sin x = 0$.
   c) $x' = t - x^2$.
   d) $7t^2x' = 3x - 2t$.
   e) $xx' = 1 - tx$.
   f) $(x')^2 + tx = \sqrt{t+1}$.

2. Find the general solution.

   a) $x' = -(2/t)x + t$.
   b) $y' + y = e^t$.
   c) $x' + 2tx = e^{-t^2}$.
   d) $tx' = -x + t^2$.
   e) $\theta' = -a\theta + \exp(bt)$.
   f) $(t^2 + 1)x' = -3tx + 6t$.

3. Solve the initial value problems.

   a) $x' + (5/t)x = 1 + t$, $x(1) = 1$.
   b) $x' = \left(a + \frac{b}{t}\right)x$, $x(1) = 1$.
   c) $R' + \frac{R}{t} = \frac{2}{1+t^2}$, $R(1) = \ln 8$.
   d) $N' = N - 9e^{-t}$, $N(0) = N_0$.
   e) $\cos\theta v' + v = 3$, $v(\pi/2) = 1$.
   f) $R' = \frac{R}{t} + te^{-t}$, $R(1) = 1$.

4. Show that the general solution to $y' + ay = \sqrt{1+t}$ is given by

$$y(t) = Ce^{-at} + \int_0^t e^{-a(t-s)}\sqrt{1+s}\,ds.$$

5. Express the general solution of the equation $x' = 2tx + 1$ in terms of the erf function.

6. Find a formula for the solution of

$$x' + \frac{e^{-t}}{t}x = t, \quad x(1) = 0.$$

7. Solve $x'' + x' = 3t$ by substituting $y = x'$.

8. Solve $x' = (t + x)^2$ by substituting $y = t + x$.

9. Find the general solution to the differential equation $x' = ax + b$, where a and b are constants, first by separation of variables, and second by integrating factors.

10. Find the general solution to the DE $x' = px + q(t)$, where p is constant. Then find the solution satisfying $x(t_0) = x_0$.

11. Consider the linear differential equation $x' + p(t)x = q(t)$, and let $x_1 = x_1(t)$ and $x_2 = x_2(t)$ be solutions. (a) Show that the sum $x(t) = x_1(t) + x_2(t)$ is a solution if, and only if, $q(t) = 0$. (b) If $x_1 = x_1(t)$ is a solution to $x' + p(t)x = 0$ and $x_2 = x_2(t)$ is a solution to $x' + p(t)x = q(t)$, show that $x(t) = x_1(t) + x_2(t)$ is a solution to $x' + p(t)x = q(t)$.

12. Show that if $x_1 = x_1(t)$ and $x_2 = x_2(t)$ are both solutions to the DE $x' + p(t)x = 0$, then x_1/x_2 is constant. Hint: Use the quotient rule for derivatives.

13. A population law is given by $N' = r(t)N - h$, where $r(t)$ is a prescribed time-dependent growth rate and h is a constant predation rate. (a) If the initial population is $N(0) = n_0$, find the solution to the population problem. (b) Plot the solution if $r(t) = \frac{1}{5}(1 + \sin t)$ and $n_0 = 8$, $h = 0.4$. Assume N is given in hundreds of animals.

14. (*Bernoulli equations*) A differential equation of the form

$$x' = a(t)x + g(t)x^n$$

is called a **Bernoulli equation**, and it arises in many important applications. Show that the Bernoulli equation can be reduced to the linear equation

$$y' = (1-n)a(t)y + (1-n)g(t)$$

by changing the dependent variable from $x = x(t)$ to $y = y(t)$ via $y = x^{1-n}$. Observe that the solution is therefore $x = y^{1/(1-n)}$.

15. Solve the Bernoulli equations:

a) $x' = \frac{2}{3t}x + \frac{2t}{x}$.

b) $x' = x(1 + xe^t)$.

c) $\theta' = -\frac{1}{t}\theta + \frac{1}{t\theta^2}$.

d) $t^2y' + 2ty - y^3 = 0$.

e) $x' = ax + bx^3, \;\; a, \; b > 0$.

f) $w' = tw + t^3w^3$.

16. (*Exact equations*). In this exercise we consider a special class of first-order differential equations called **exact equations**, which occur in many applications. They have the form

$$f(t,x) + g(t,x)x' = 0, \tag{1.18}$$

where the left side has the form of a total derivative. That is, there is a function $H = H(t,x)$ for which

$$\frac{d}{dt}H(t,x) = f(t,x) + g(t,x)x'.$$

From the chain rule the total derivative of $H(t,x)$ is also

$$\frac{d}{dt}H(t,x) = H_t(t,x) + H_x(t,x)x'.$$

Therefore, if $H_t = f$ and $H_x = g$, then the differential equation is exact. Then the differential equation can be written $\frac{d}{dt}H(t,x) = 0$, which implies $H(t,x) = C$, for some arbitrary constant C. Therefore the integral curves, or solution of (1.18) is given implicitly by $H(t,x) = C$.

a) Show that $f(t,x) + g(t,x)x' = 0$ is an exact equation if $f_x = g_t$. (This condition is also implied by exactness.)

b) Use part (a) to check if the following equations are exact. If the equation is exact, find the general solution by solving $H_t = f$ and $H_x = g$ for H. (You may want to review the method of finding potential functions associated with a conservative force field from your multivariable calculus course.) Write down the solution, or integral curves.

i) $x^3 + 3tx^2x' = 0$.

ii) $t^3 + \frac{x}{t} + (x^2 + \ln t)x' = 0$.

iii) $x' = -\frac{\sin x - x \sin t}{t \cos x + \cos t}$.

iv) $x^3 dt + 3tx^2 dx = 0$.

v) $x^2 dt - t^2 dx = 0$.

v) $t(\cot x)x' = -2$.

17. (*Orthogonal trajectories*) Solutions to a first-order equation form a set of integral curves of the form $\phi(x, y) = C$, where C is a constant. (Here we are using x and y as variables because the topic is geometry in the plane.) The integral curves, for example, may be the equipotential curves of a conservative force field, such as an electric field. Every set of integral curves is defined by a differential equation $\phi_x dx + \phi_y dy = 0$, or $dy/dx = -\phi_x/\phi_y$. We often want to find the integral curves that are perpendicular, or orthogonal, to the given curves; these are called the **orthogonal trajectories**. For example, we may want to find the flux lines of the electric field. Because orthogonal families of curves are perpendicular, their slopes are negative reciprocals. Therefore, the integral curves orthogonal to $\phi(x, y) = C$ satisfy the differential equation
$$\frac{dy}{dx} = \frac{\phi_y}{\phi_x}.$$
Plot the following integral curves and then find and plot the orthogonal trajectories.

a) $x^2 + y^2 = C$.

b) $xy = C$.

c) $y = Cx^2$.

1.4.2 Applications

Now we consider some practical examples of linear models, which are myriad. Additional ones are given in the exercises.

Example 1.26

(**Newton's Law of Cooling**) When the environmental temperature Q is not constant, but rather $Q(t)$, a function of time, then Newton's law of cooling becomes
$$T' = -h(T - Q(t)), \quad T(0) = T_0.$$
This equation can be rearranged and written in the form
$$T' + hT = hQ(t),$$

which is in the standard form of a first-order linear equation. The reader should verify that the general solution is

$$T(t) = Ce^{-ht} + e^{-ht} \int_0^t hQ(s)e^{hs} ds. \quad \square$$

Example 1.27

(**Chemical reactor**) The general equation governing the concentration $C(t)$ in a chemical reactor with variable volumetric flow rates q_{in} and q_{out} is

$$(V(t)C)' = q_{\text{in}} C_{\text{in}} - q_{\text{out}} C,$$

where $V(t) = V_0 + (q_{\text{in}} - q_{\text{out}})t$ is the volume of mixture in the reactor. This equation is linear because it can be put in the form (show this!)

$$C' + \left(q_{\text{out}} + \frac{V'(t)}{V(t)} \right) C = \frac{1}{V(t)} q_{\text{in}} C_{\text{in}}. \quad \square$$

Example 1.28

(**Sales Response to Advertising**) The field of economics is a rich source of interesting phenomena modeled by differential equations. In this example we set up a simple model that allows management to assess the effectiveness of an advertising campaign. Let $S = S(t)$ be the monthly sales of an item. In the absence of advertising it is observed from sales history data that sales decay over time. Thus $S' = -aS$, where a is the decay constant. To keep sales up, advertising is required. If there is a lot of advertising, then sales tend to saturate at some maximum value $S = M$; this is because there are only finitely many consumers; then $M - S$ is the untapped market. The rate of increase in sales due to advertising is jointly proportional to the advertising rate $A(t)$ and to the degree the market is not saturated; that is,

$$rA(t) \left(\frac{M - S}{M} \right).$$

The constant r measures the effectiveness of the advertising campaign. The term $(M - S)/M$ is a measure of the market share that has still not purchased the product. Then, combining both natural sales decay and advertising, we obtain the economic model

$$S' = -aS + rA(t) \left(\frac{M - S}{M} \right).$$

The first term on the right is the natural decay rate, and the second term is the rate of sales increase due to advertising, which drives the sales. We can rearrange the terms and write the equation in the form of a linear equation

$$S' = -\left(a + \frac{rA(t)}{M}\right) S + rA(t). \quad \square \tag{1.19}$$

EXERCISES

1. (*Mixtures*) Initially, a tank contains 60 gal of pure water. Then brine containing 1 lb of salt per gallon enters the tank at 2 gal/min. The perfectly mixed solution is drained off at 3 gal/min. Determine the amount (in lbs) of salt in the tank up until the time it empties.

2. (*Economics*) Determine the units of the various quantities in the sales–advertising model (1.19) (e.g., S is measured in dollars). If A is constant, what is the sales equilibrium?

3. (*Technology transfer*) Suppose a new innovation is introduced at time $t = 0$ in a community of N possible users (e.g., a new pesticide introduced to a community of farmers). Let $x(t)$ be the number of users who have adopted the innovation at time t. If the rate of adoption of the innovation is jointly proportional[2] to the number of adoptions and the number of those who have not adopted, write down a DE model for $x(t)$. Describe, qualitatively, how $x(t)$ changes in time. Find a formula for $x(t)$.

4. (*Home heating*) A house is initially at 12 degrees Celsius when its heating–cooling system fails. The outside temperature varies according to $Q(t) = 9 + 10\cos 2\pi t$, where time is given in days. The heat loss coefficient is $h = 3$ degrees per day. Find a formula for the temperature variation in the house and plot it along with $Q(t)$ on the same set of axes. What is the time lag between the maximum inside and outside temperature?

5. (*Economics*) Let $M(t)$ be the total amount of money a household possesses at time t. If they spend money at a rate proportional to how much money they have, and $I(t)$ is their income, or the rate they earn money, set up a model for the total amount of money on hand. Assume $M(0) = m_0$ and show that

$$M(t) = m_0 e^{-at} + e^{-at}\int_0^t I(s)e^{as}ds.$$

Use l'Hospital's rule to find the limiting of $M(t)$ as $t \to \infty$.

[2] A quantity Q is jointly proportional to quantities A and B if $Q = kAB$ for some constant k.

6. (*Advertising*) In the sales response to advertising model (1.19), assume $S(0) = S_0$ and that advertising is constant A over a fixed time period T, and is then removed. That is,

$$A(t) = \begin{cases} A, & 0 \le t \le T \\ 0, & t > T \end{cases}$$

Find a formula for the sales $S(t)$. Hint: Solve the problem on two intervals and piece together the solutions in a continuous way.

7. (*Mechanics*) An object of mass $m = 1$ is dropped from rest at a large height, and as it falls it experiences the force of gravity mg and a time-dependent resistive force of magnitude $F_r = 2v/(t+1)$, where v is its velocity. Write down an initial value problem that governs its velocity and find a formula for the solution.

8. (*Species migration*) The MacArthur–Wilson model of the dynamics of species (e.g., bird species) that inhabit an island located near a mainland was developed in the 1960s. Let P be the constant number of species in the source pool on the mainland, and let $S = S(t)$ be the number of species on the island. Assume that the rate of change of the number of species is

$$S' = \chi - \mu,$$

where χ is the colonization rate and μ is the extinction rate. In the MacArthur–Wilson model,

$$\chi = I\left(1 - \frac{S}{P}\right) \quad \text{and} \quad \mu = \frac{E}{P}S,$$

where the constants I and E are the maximum colonization and extinction rates, respectively.

   a) Over a long time, what is the expected equilibrium for the number of species inhabiting the island?

   b) Given $S(0) = S_0$, find an analytic formula for $S(t)$.

   c) Suppose there are two islands, one large and one small, with the larger island having the smaller maximum extinction rate. Both have the same colonization rate. Show, as expected, that the smaller island eventually has fewer species.

9. (*Mortality*) Let N_0 be the number of people born on a given day (a cohort), and assume they die at the per capita rate $m(t)$, where t is their age.

a) Find the number of individuals $N(t)$ remaining in the cohort at age t. The fraction of the cohort that lives to age t is $S(t) = N(t)/N_0$ and is called the *survivorship function*. What is the probability that a member of the cohort dies before age t?

b) What is the probability of dying between the ages of $t = a$ and $t = b$?

c) The Weibull model of mortality is defined by

$$m(t) = \frac{p+1}{p_0}\left(\frac{t}{t_0}\right),$$

where p_0, t_0, and p are parameters. Find $S(t)$ for $p = 0$, $p = 3$, and $p = 10$. Which one seems to best fit the human population? A fish population?

10. (*Reactors*) A chemical flows into a reactor at concentration C_{in} with volumetric flow rate q. While in the reactor it chemically reacts according to $\mathbf{C} + \mathbf{C} \to$ Products. The mixture flows out at the same rate q. The governing equation for the concentration of $\mathbf{C}$ is (see Section 1.7)

$$(VC)' = qC_{\text{in}} - qC - kVC^2.$$

Initially, take $C(0) = C_0$ and assume $C_{\text{in}} = 0$. Show that this is a Bernoulli equation and solve it.

11. (*Parasite infections*) One study on the effect of a parasitic infection on an animal's immune system was carried out with the intestinal nematode parasite *Heligmosoides polygyrus* and a fixed number of laboratory mice. Mice were fed parasite larva at the constant rate of λ larva per mouse, per day. The larva migrate to the wall of the small intestine. There they die at per capita rate μ_0, and they develop into mature parasites, which migrate to the gut lumen, at the per capita rate of μ. The mature parasites die at the per capita rate δ. If $L = L(t)$ is the average number of larva per mouse, and $M = M(t)$ is the average number of mature parasites per mouse, then the model becomes

$$L' = \lambda - (\mu_0 + \mu)L,$$
$$M' = \mu L - \delta M.$$

Initially, $L(0) = M(0) = 0$. First, explain the terms in the model. Then solve the larva equation and substitute the solution into the mature parasite equation to find $M(t)$. Make generic plots of L and M vs. t. [For more details regarding the experiment, the constants, and the immune response, see J. D. Murray, 2002. *Mathematical Biology I. An Introduction*, 3rd ed., Springer, New York, pp. 351–361.]

12. (*Population growth*) Find the general solution of each of the two general forms of the logistic equation,

$$X' = r(t)X\left(1 - \frac{X}{K}\right),$$

and

$$X' = X\left(1 - \frac{X}{K(t)}\right).$$

Answers should be in terms of indefinite integrals. Hint: Bernoulli.

13. (*Groundwater contamination*) (Brennan & Boyce, 2011, p. 118) When chlorinated solvents, such as trichloroethylene, are spilled or dumped on a land surface they leach downward into subsurface groundwater and contaminate it. They are denser than water and only slightly soluble, they remain more or less stationary in the flowing groundwater, forming a toxic source. As pure groundwater flows through the contamination region, it picks up toxic particles and carries them downstream. The diagram in Figure 1.9 gives a visual representation of the contamination event. We assume the stationary contaminant source is a cube with cross-sectional area A, and the groundwater speed is V (called the Darcy velocity); $M(t)$ is the mass of the contaminant in the source region and $C(t)$ is the concentration (mass per volume) of the contaminant leaving the source region. The rate of change of the mass M is given by

$$\frac{dM}{dt} = -AVC.$$

The concentration C clearly depends on M. A constitutive relation is determined experimentally, and one common assumption is

$$\frac{C}{c_0} = \left(\frac{M}{m_0}\right)^{\beta}, \quad \beta > 0,$$

where m_0 is the initial mass of contaminant in the source and c_0 is the resulting initial source region concentration. In summary we obtain an equation that models the dissolution of the mass in the source region caused by groundwater flow:

$$\frac{dM}{dt} = -\alpha M^{\beta}.$$

Finally, degradation due to biotic and abiotic processes causes decay and we have

$$\frac{dM}{dt} = -\alpha M^{\beta} - rM. \tag{1.20}$$

a) Show $\alpha = VAc_0/m_0^{\beta}$.

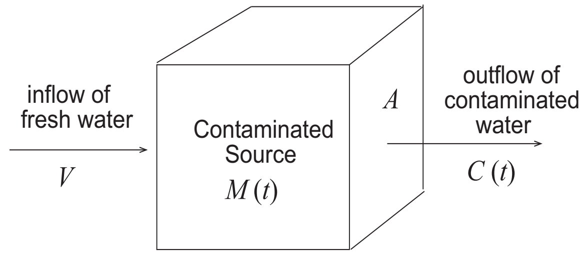

Figure 1.9 Representation of a subsurface contaminant source. M is the total mass of contaminant in the source, C is the concentration of the dissolved contaminant leaving the source, and V is the groundwater velocity.

b) Solve equation (1.20) with $M(0) = m_0$ in the cases (i) $\beta = 1$, (ii) $\beta \neq 1$. Note $M(0) = m_0$.

c) Let $0 < \beta < 1$. Does the contaminant source have a finite lifetime? (Consider both cases, $r = 0$ and $r > 0$.)

d) Take $m_0 = 1620$ kg, $c_0 = 100$ mg/L, $A = 30$ m^2, $V = 20$ m/year, and $r = 0$. Plot a graph of $M(t)$ in the cases $\beta = 2$ and $\beta = 0.5$ for $0 \leq t \leq 100$ years.

1.4.3 Electrical Circuits

Modern technological society is filled with electronic devices of all types. At the base of these are electrical circuits. The simplest circuit unit is the loop in Figure 1.10 that contains an electromotive force (emf) $E(t)$ (a battery or generator that supplies energy), a resistor, an inductor, and a capacitor, all connected in series. A capacitor *stores electrical energy* on its two plates, a resistor *dissipates energy*, usually in the form of heat, and an inductor *stores energy in its magnetic field*, which resists changes in current; we think of it as a coil. A basic law in electricity, **Kirchhoff's law,** states that the sum of the voltage drops across the circuit elements (as measured, e.g., by a voltmeter) in a loop must equal the applied emf. In symbols,

$$V_L + V_R + V_C = E(t).$$

This law comes from conservation of energy in a current loop, and it is derived in elementary physics and engineering texts. A voltage drop across an element is an energy potential that equals the amount of work required to move a charge across that element.

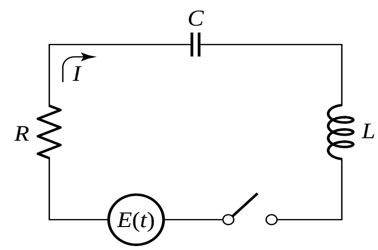

Figure 1.10 An RCL circuit with an electromotive force $E(t)$ supplying the electrical energy.

Current in a circuit is analogous to water flowing in a pipe. In this water analogy, the voltage drop is the difference in water pressure between two points along the pipe (measured by a pressure meter, say), and particularly the pressure drop across an element. Water flowing in the pipe is measured as the rate that water passes a fixed point. A resistor is like a mesh in a pipe, taking more energy to push water through it. A capacitor is like a tank that stores water, and an inductor is like a water wheel at a old mill, which because of its inertia, resists changes in its angular velocity when the water flow is increased. There are enlightening articles on the internet about the electrical–water analogy.

Let $I = I(t)$ denote the current (in amperes, or charge per second) in the circuit, and let $Q = Q(t)$ denote the charge (in coulombs) on the capacitor. These quantities are related by

$$Q' = I.$$

By **Ohm's law** the voltage drop across the resistor is proportional to the current, or

$$V_R = RI,$$

where the proportionality constant R is called the resistance (measured in ohms). The voltage drop across a capacitor is proportional to the charge on the capacitor, or

$$V_C = \frac{1}{C}Q,$$

where C is the capacitance (measured in farads). Finally, the voltage drop across an inductor is proportional to how fast the current is changing, or

$$V_L = LI',$$

where L is the inductance (measured in henrys). This is **Faraday's law**. Substituting these voltage changes into Kirchhoff's law gives

$$LI' + RI + \frac{1}{C}Q = E(t),$$

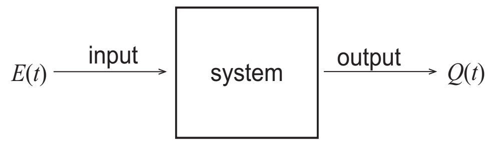

Figure 1.11 An input-output system.

or, using $Q' = I$,

$$LQ'' + RQ' + \frac{1}{C}Q = E(t). \quad \text{(RCL circuit)} \tag{1.21}$$

This is the **RCL circuit equation**, which is a second-order differential equation for the charge $Q = Q(t)$ on the capacitor. The initial conditions are

$$Q(0) = q_0, \quad Q'(0) = I(0) = I_0.$$

These initial conditions express the initial charge on the capacitor and the initial current in the circuit. Here, $E(t)$ may be a prescribed constant (e.g., $E(t) = 12$ for a 12-volt battery) or it may be a oscillating function of time t (e.g., $E(t) = A\cos\omega t$ for an alternating voltage potential of amplitude A and frequency ω).

Equation (1.21) can be written in terms of the current I as

$$LI'' + RI' + \frac{1}{C}I = E'(t).$$

Equation (1.21) is a second-order equation and is examined in Chapter 2. However, if there is no inductor in the circuit, then the resulting RC circuit is modeled by the first-order equation

$$RQ' + \frac{1}{C}Q = E(t). \quad \text{(RC circuit)}$$

If $E(t)$ is constant, for example for a battery, this equation can be solved using separation of variables. Otherwise, we must divide the equation by R to put into the standard form of a linear equation,

$$Q' + \frac{1}{RC}Q = \frac{1}{R}E(t),$$

which can be solved using integrating factors.

From an engineering perspective, circuits illustrate a *systems* view of problems. Regarding the RC circuit as a system, we **input** the voltage source $E(t)$ and get the charge $Q(t)$ as the **output**. This is illustrated in Figure 1.11.

Example 1.29

Let $R = 1$; $C = 1/2$, and $Q(0) = 0$, with an emf given by

$$E(t) = \begin{cases} 1, & 0 \le t < 2, \\ 0, & t > 2. \end{cases}$$

For $t < 2$ the circuit equation is $Q_1' + 2Q_1 = 1$, and for $t > 2$ the equation is $Q_2' + 2Q_2 = 0$. These have solutions

$$Q_1(t) = \frac{1}{2}\left(1 - e^{-2t}\right), \ t < 2, \qquad Q_2(t) = Ae^{-2t}, \ t > 2,$$

respectively, where the initial condition has been applied to the Q_1 equation; A is an arbitrary constant. Because the solution is continuous, we impose $Q_1(2) = Q_2(2)$, which gives $A = (1/2)(e^4 - 1)$. So

$$E(t) = \begin{cases} \frac{1}{2}\left(1 - e^{-2t}\right), & 0 \le t \le 2, \\ \frac{1}{2}\left(e^4 - 1\right)e^{-2t}, & t > 2. \end{cases} \qquad \square$$

EXERCISES

1. Write down the equation that governs an RC circuit with a 12-volt battery, taking $R = 1$ and $C = \frac{1}{2}$. Determine the equilibrium solution and its stability. If $Q(0) = 5$, find a formula for $Q(t)$. Find the current $I(t)$. Plot the charge and the current on the same set of axes.

2. An aging battery generating $200e^{-5t}$ volts is connected in series with a 20 ohm resistor, and a 0.01 farad capacitor. Assuming $q = 0$ at $t = 0$, find the charge and current for all $t > 0$. Show that the charge reaches a maximum and find the time it is reached.

3. In an arbitrary RC circuit with constant emf E, use the method of separation of variables to derive the formula

$$Q(t) = Ae^{-t/RC} + EC$$

   for the charge on the capacitor, where A is an arbitrary constant. If $Q(0) = Q_0$, what is A?

4. An RCL circuit with an applied emf given by $E(t)$ has initial charge $Q(0) = Q_0$ and initial current $I(0) = I_0$. What is $I'(0)$? Write down the circuit equation and the initial conditions in terms of current $I(t)$. Hint: Use Kirchhoff's law in the form $LI' + RI + Q/C = E(t)$.

5. Write the RCL circuit equation with the voltage $V_c(t)$ on the capacitor as the unknown state function.

6. Formulate the governing equation of an RCL circuit in terms of the current $I(t)$ when the circuit has an emf given by $E(t) = A\cos\omega t$. What are the appropriate initial conditions?

7. Find the DE model for the charge in an LC circuit with no emf. Show that the response (or, solution) of the circuit can have the form $Q(t) = A\cos\omega t$ for some amplitude A and frequency ω, both of which are determined in terms of the circuit parameters L and C.

8. Consider a standard RCL circuit with no emf, but with a voltage drop across the resistor given by a nonlinear function of current,

$$V_R = \frac{1}{2}\left(\frac{1}{3}I^3 - I\right)$$

   Such a resistor is called an *active* resistor, in contrast to Ohm's law, which describes a *passive* resistor. If $C = L = 1$, find a second-order differential equation for the current $I(t)$ in the circuit.

9. Use separation of variables to solve the following problems. Write the solution explicitly when possible.

   a) $x' = p(t)x$, where $p(t)$ is a given continuous function.

   b) $x' = -2tx, \quad x(1) = 2$. Plot the solution on $0 \le t \le 2$.

   c) Let

$$x' = \begin{cases} -2x, & 0 < t < 1 \\ -x^2, & 1 \le t \le 2, \end{cases}$$

   with $x(0) = 5$. Find and plot a continuous solution on the interval $0 \le t \le 2$.

1.5 One-Dimensional Dynamical Systems

So far we focused on setting up and solving differential equations with the goal of finding an analytical solution, or formula. In this section we take a *qualitative* approach and see how important information can be obtained using geometric methods. Often, in science and engineering, qualitative methods give all the information we need to understand the problem without deriving detailed and complicated formulas.

Many applications lead to differential equations that have no *explicit* time dependence; that is, the rate of change depends only on the state $x = x(t)$.

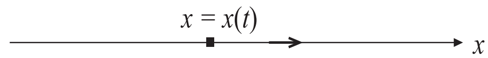

Figure 1.12 The phase line. A one-dimensional state space x where the state of the system $x = x(t)$ moves as t changes. Arrows on the line represent the direction of motion, as shown.

Such an equation has the form

$$\frac{dx}{dt} = f(x),$$

with no *explicit* time dependence t on the right side. (Of course, the state itself x is time dependent.) Such equations are called **autonomous**. A common representation of a solution $x = x(t)$ is a time series plot of x versus t in the two-dimensional tx plane. But here we interpret a solution as a *state* x moving along a one-dimensional line, the x axis. See Figure 1.12. The x axis is a state space called the **phase line**. Think of a particle of mass m moving on the x axis, which is the position of the particle. For comparison, in multivariable calculus we regarded of the motion of a particle moving in the xy plane along a given trajectory, for example, counterclockwise movement on a circle $x^2 + y^2 = R^2$ of radius R. That is an important piece of information even though we may not know the explicit parametric equations $x = x(t)$, $y = y(t)$ of the trajectory. We give up knowledge of time dependence and focus upon directions.

1.5.1 Autonomous Equations

Example 1.30

(**The Logistic Law**) In animal populations we do not expect exponential growth over long times as predicted by the Malthus law where the relative growth rate $x'/x = r$ remains constant as the population x increases. Environmental factors such as competition for resources limit the population as it gets large. Therefore we expect the relative growth rate r to decrease as x gets large. The simplest assumption is that the relative growth rate decreases *linearly* as a function of population, and the rate becomes zero at some maximum *carrying capacity* K. See Figure 1.13. This is the **logistic model** of population growth, developed by P. Verhulst in the 1800s. Quantitatively,

$$\frac{x'}{x} = r\left(1 - \frac{x}{K}\right) \quad \text{or} \quad x' = rx\left(1 - \frac{x}{K}\right). \tag{1.22}$$

We can write this autonomous equation in the form

$$x' = rx - \frac{r}{K}x^2.$$

The first term is a positive *growth term*, which is the Malthus term. The second term, which is a negative quadratic in x, decreases the population growth rate and is the *competition term*. The latter is a reasonable model because if there were x animals, then there would be about x^2 encounters; so the competition term is proportional to the number of possible encounters. For any

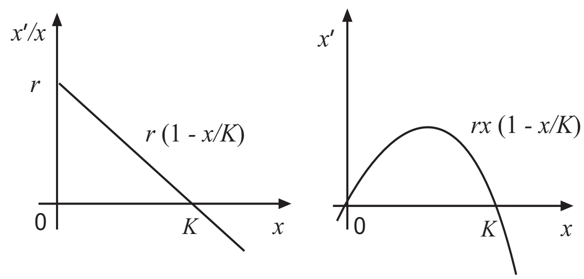

Figure 1.13 Plots of the logistic model of population growth. The left plot shows the relative growth rate versus population, and the right plot shows the growth rate versus population. Either plot gives an important interpretation of the model. Where the growth rates are above the axis, the population increases; when either is below the axis, the population decreases.

initial condition $x(0) = x_0$ we can find the formula for the solution to the logistics equation (1.22) using separation of variables. However, qualitative features of solutions can be exposed without actually finding it. We often seek only qualitative features of a model. We note there are two constant solutions to (1.22),

$$x(t) = 0, \qquad x(t) = K \qquad \text{(equilibrium solutions)}$$

corresponding to no animals (extinction) and corresponding to the number of animals at the carrying capacity K. These constant solutions are found by setting the right side of the growth equation equal to zero (because that forces $x' = 0$, or $x =$ constant. Constant solutions to an autonomous equations are called **steady-state**, or **equilibrium**, solutions. Further:

1. If the population is between $x = 0$ and $x = K$ then the right side of (1.22) is positive, giving $x' > 0$; for these population numbers, the population is increasing.

2. If the population is larger than the carrying capacity K, then the right side of (1.22) is negative, giving $x' < 0$, and thus the population is decreasing.

These facts can be represented geometrically on a plot of the growth rate x' versus the population x as shown in Figure 1.13. The key properties can be represented conveniently on a **phase line** plot as shown in Figure 1.14. We first plot the growth rate x' versus x, which in this case is a parabola opening downward (Figure 1.13). The points of intersection on the x axis are the equilibrium solutions $x^* = 0$ and $x^* = K$. We then indicate by a directional arrow on the x axis those values of x where the solution $x(t)$ is increasing (where $x' = f(x) > 0$) or decreasing ($x' = f(x) < 0$). The arrow points to the right when the graph of the growth rate $f(x)$ is above the axis, and it points to the left when the graph is below the axis. We view the phase line as a one-dimensional, parametric solution space with the population $x = x(t)$ tracing out points on that line as t increases. In the range $0 < x < K$ the arrow points right because $x' > 0$. So $x(t)$ increases in this range. For $x > K$ the arrow points left because $x' < 0$. The population $x(t)$ decreases in this range. Most of the time, rather than draw the phase line directly below the plot of the growth rate, as shown, we just draw the arrows on the x axis of the growth rate plot.

These qualitative features can easily be transferred to approximate time series plots (see Figure 1.15) showing $x(t)$ versus t for different initial conditions. □

Both the phase line and the time series plots imply that, regardless of the initial population (if nonzero), the population approaches the carrying capacity K. This equilibrium population $x^* = K$ is called a **stable equilibrium**. The zero population $x^* = 0$ is also an equilibrium population; but, near zero we have $x' > 0$, and so the population diverges away from zero. We say the equilibrium population $x^* = 0$ is an **unstable equilibrium**. Because we consider only nonnegative populations, we ignore the fact that $x = 0$ could be approached on the left side. In summary, this simple geometrical analysis determines the complete qualitative structure of the logistic population model.

Example 1.31

(**Motion in a Fluid**) Suppose a particle of mass m is falling downward through a viscous fluid. There are two forces on the particle, gravity and fluid resistance. If we measure positive distance downward from the top of the fluid surface, the gravitational force is mg and is positive because it acts on the mass in the positive downward direction; the resistive force is $-av^2$, and it is negative because it opposes positive downward motion. The net external force is then

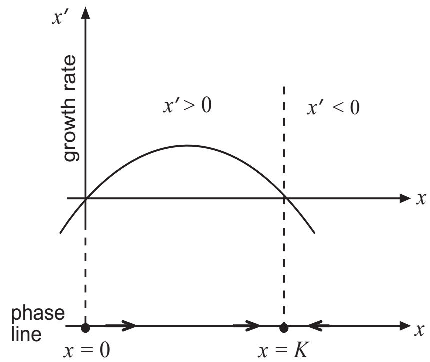

Figure 1.14 The x axis is the phase line, on which arrows indicate an increasing or decreasing population for certain ranges of x.

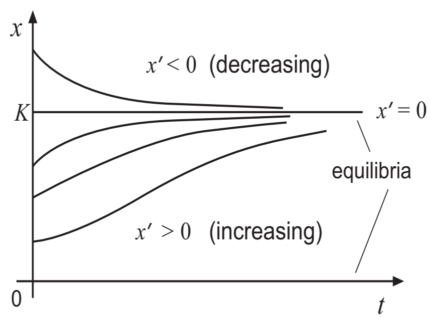

Figure 1.15 Time series plots of solutions to the logistics equation for various initial conditions. For $0 < x < K$ the population increases and approaches K, whereas for $x > K$ the population decreases to K. If $x(0) = K$, then $x(t) = K$ for all times $t > 0$; this is the equilibrium solution.

$F = mg - av^2$, and the equation of motion, from Newton's second law, is

$$mv' = mg - av^2.$$

If we impose an initial velocity, $v(0) = v_0$, then the differential equation and initial condition give an initial value problem for $v = v(t)$ which can be solved by separation of variables.

As in the last example we can obtain important qualitative information from the DE itself, without solving. Over a long time, assuming the fluid is deep, we would observe that the falling mass would approach a constant *terminal velocity* v_T. Physically, the terminal velocity occurs when the two forces, the

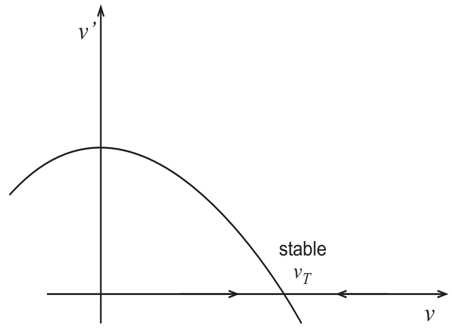

Figure 1.16 A phase line plot for the model $v' = g - (a/m)v^2$. For $v < v_T$ the velocity is increasing because $v' > 0$; for $v > v_T$ the velocity is decreasing because $v' < 0$. All the solution curves approach the equilibrium solution, which is the constant terminal velocity $v(t) = v_T$.

gravitational force and resistive force, balance, thus giving $v' = 0.$, or $g - (av_T^2/m) = 0$. Therefore,

$$v_T = \sqrt{\frac{mg}{a}}.$$

Note that $v(t) = v_T$ is the constant, or equilibrium, solution of the differential equation. Regardless of the initial velocity, the system approaches this equilibrium state. This supposition is supported by the observation that $v' > 0$ when $v < v_T$ and $v' < 0$ when $v > v_T$. Figure 1.16 shows the phase line diagram indicating the approach to equilibrium. $\square$

General Discussion. Qualitative methods can be used to examine any autonomous equation,

$$x' = f(x). \tag{1.23}$$

An **equilibrium solution** of (1.23) is a constant solution $x(t) = x^*$ for all t. Clearly, those solutions have zero derivative and therefore must be roots of the algebraic equation $f(x) = 0$. Thus, if $x(t) = x^*$ is an equilibrium, then $f(x^*) = 0$, and conversely. The roots of $f(x) = 0$ can be regarded as points on the x axis, and therefore we call them **critical points** of f; in our language, we usually make no distinction between a critical point and the corresponding equilibrium solution. Critical points are the values where the graph of $f(x)$ versus x intersects the x axis. We always assume the critical points are **isolated**; that is, if x^* is a critical point, then there is a small open interval containing x^* that contains no other critical points. Figure 1.17 shows a generic plot where the critical points are $x^* = a$, b, c. In between the equilibria we can note the values of x for which the population is increasing ($f(x) > 0$) or decreasing ($f(x) < 0$). We place arrows on the phase line, or the x-axis, in between the critical points showing the direction of the movement (increasing or decreasing) of $x(t)$ as time increases. If desired, the information from the phase line can be translated into time series plots of $x(t)$ versus t (Figure 1.18). In between the constant, equilibrium solutions, the other solution curves increase or decrease; clearly, oscillations are not possible. Moreover, assuming f is a well-behaved function ($f'(x)$ is continuous), solution curves actually approach some equilibria, getting closer and closer as time increases. Because solutions are unique (Section 1.5.3), two solution curves can never intersect.

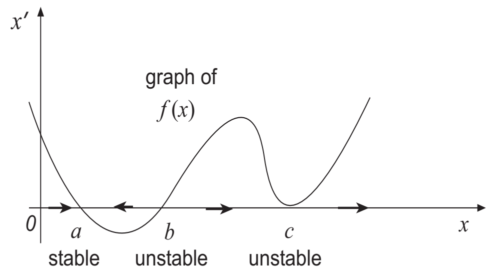

Figure 1.17 A generic phase line diagram showing $f(x)$, which is x' versus x. The points of intersection, a, b, c, on the x-axis are the equilibria, or critical points. The arrows on the x-axis, which is the phase line, show how the state x changes with time between the equilibria. The direction of the arrows is easily read from the plot of $f(x)$ vs. x. They are to the right when $f(x) > 0$ and to the left when $f(x) < 0$.

On the phase line (x axis), if arrows on both sides of a critical point toward that critical point, then we say the critical point, or equilibrium, is **stable**, or sometimes asymptotically stable. If both of the arrows point away, the critical point or equilibrium is **unstable**. If a direction arrow on one side of a critical points is directed toward the the critical point, and on the other side it is directed away, or vice-versa, then we say the critical point is unstable; some authors use the term *semi-stable*.

Our interpretation is this. If a system is in a stable equilibrium and given a small **perturbation** (i.e., a small change, or "bump") moving it to a nearby state, then the system returns to that state as $t \to +\infty$. Specifically, consider a population of fish in a lake that is in an stable state x^*. A small death event, say caused by some toxic chemical that is dumped into the lake, will cause the population to drop. Stability means that the system will return the original state x^* over time. At an unstable equilibrium, a small perturbation can cause

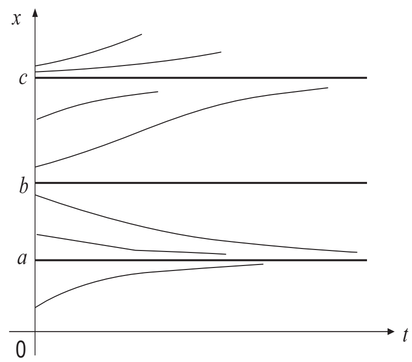

Figure 1.18 Time series plots corresponding to Figure 1.17 for different initial conditions. The constant solutions are the equilibria. The equilibrium $x = a$ is *stable*—if the system is in that state and given a small perturbation away from that state, the system will return to $x = a$. Both $x = b$ and $x = c$ are unstable equilibria.

the system to approach a different equilibrium or even go off to infinity. In the logistics model for population growth we observed that $x = K$ is stable; if a small change from the carrying capacity occurs, the system will return to $x = K$. The zero population $x = 0$ is unstable; a small perturbation to zero will cause the number of fish to grow and approach the carrying capacity $x = K$ at $t \to +\infty$.

Importantly, when we say an equilibrium x^* is stable, we understand it is with respect to *small perturbations*, or in a local sense. A more precise definition of stability can be given as follows. An isolated equilibrium state x^* of (1.23) is *locally stable* if there is an open interval I containing x^* with $\lim_{t\to+\infty} x(t) = x^*$ for any solution $x = x(t)$ of (1.23) with $x(0)$ in I. That is, each solution starting in I converges to x^*.

If a perturbation is not small, there is no guarantee that the system returns to the original stable state x^*. For example, a catastrophe or bonanza in a fish population could cause the population to jump beyond some other equilibrium. If the population returns to the state x^* for *all* perturbations, no matter how large, then the state x^* is called **globally stable**.

Analytic Criterion. We can write a simple analytic criterion for stability of an equilibrium. From the phase line plot, if x^* is an equilibrium and the derivative $f'(x^*) < 0$ ($f(x)$ is decreasing), then the pattern of arrows on the phase line is that of a stable equilibrium, both pointing toward x^*. Similarly, if $f'(x^*) > 0$ ($f(x)$ is increasing) then the arrows on the phase line point away from x^*, and the equilibrium is unstable. If $f'(x^*) = 0$, then all patterns are possible and the equilibrium may be stable or unstable. In this latter case the concavity of $f(x)$ at x^*) comes into play and we must examine the second derivative to determine concavity. We summarize this discussion in the following:

Theorem 1.32

Let x^* be an isolated critical point, or equilibrium, for the autonomous equation

$$\frac{dx}{dt} = f(x).$$

If $f'(x^*) < 0$, then x^* is stable; if $f'(x^*) > 0$, then x^* is unstable. If $f'(x^*) = 0$, then there is no information about stability. In this case we analyze higher derivatives. □

Example 1.33

(**Logistic Equation**) Consider the logistics equation $x' = f(x) = rx(1-x/K)$. The equilibria are $x^* = 0$ and $x^* = K$. The derivative of $f(x)$ is

$$f'(x) = r - 2rx/K.$$

Evaluating the derivative at the equilibria gives

$$f'(0) = r > 0, \quad f'(K) = -r < 0.$$

Therefore $x^* = 0$ is unstable and $x^* = K$ is stable. □

In summary, an autonomous model can be quickly and easily analyzed qualitatively without finding the solution. All we do is plot $f(x)$ versus x, and then identify on the phase line the equilibria and their stability properties.

Remark 1.34

For autonomous equations of the form $x' = F(x) - G(x)$, where the right cannot be factored, we can find the equilibria graphically by plotting $F(x)$ versus x and $G(x)$ versus x on the same set of axes. The points x^* where the graphs cross, i.e., where $F(x^*) = G(x^*)$, are the equilibria. Where $F(x) > G(x)$, the state $x = x(t)$ is increasing, and where $F(x) < G(x)$, the state $x = x(t)$ is decreasing. □

EXERCISES

1. A fish population in a lake is harvested at a constant rate, and it grows logistically. The growth rate is 0.2 per month, the carrying capacity is 40 (thousand), and the harvesting rate is 1.5 (thousand per month). Write down the model equation, find the equilibria, and classify as stable or unstable. Will the fish population ever become extinct? What is the most likely long-term fish population?

2. For each of the following models: (a) plot the growth rate $f(x)$ versus x and sketch the phase line diagram; (b) find the equilibria analytically and classify them according to their stability using Theorem 1.32; (c) draw a few key time series plots, $x = x(t)$ in the tx plane.

   a) $x' = x^2(2-x)$.
   b) $x' = x(4-x)(5-x)^2$.
   c) $x' = (x-1)e^{-2x}$.
   d) $x' = x(x-8)^3$.
   e) $x' = 2x - 7$.
   f) $x' = 2x(1-x) - \frac{1}{2}x$.
   g) $x' = (4-x)(2-x)^3$.
   h) $x' = x^2(5-x)^2(x-10)$.
   i) $x' = -(1+x)(x^2-4))$.
   j) $x' = \cosh x - 1$.

3. Use analytical or graphical methods to determine equilibria for each of the following differential equations:

   a) $x' = (1-x)\left(1-e^{-2x}\right)$.
   b) $y' = y^4\left(1-ye^{-ay}\right),\ a > 0$.
   c) $R' = \frac{3R}{1+R^2} - 1$.
   d) $z' = \frac{1}{a^2+z} - \ln z$.

4. (*The Allee effect*) At low population densities it may be difficult for an animal to reproduce because of a limited number of suitable mates. A population model that predicts this behavior is the Allee model (W. C. Allee, 1885–1955)

$$P' = rP\left(\frac{P}{a} - 1\right)\left(1 - \frac{P}{K}\right), \quad 0 < a < K.$$

   (a) Find the growth rate P' versus P and sketch the phase line diagram. (b) Determine the equilibrium populations analytically. (c) Classify the equilibria with respect to stability. (d) Describe the long time behavior of the system for different initial populations. (e) Comment on the meaning of the model.

5. Consider the autonomous DE

$$\frac{dx}{dt} = (x^2 - 36)(a - x)^2,$$

where $a > 10$ is a constant. (a) Draw the phase line diagram. (b) Sketch an approximate graph of the solution curve $x = x(t)$ satisfying $x(0) = 8$.

6. (*Heat transfer*) Heat transfer by radiation from a body to its surroundings is modeled by the Stefan–Boltzmann law

$$\frac{dT}{dt} = -k\left(T^4 - S^4\right),$$

where $T = T(t)$ is the absolute temperature of the body and S is the temperature of the surroundings. k is a physical parameter depending on the body. (a) Draw a phase line diagram and comment on the nature of the solution. (b) Let $T(0) = 2000\,\mathrm{K}$ and $S = 300\,\mathrm{K}$. Since S is much smaller than T we can neglect the term S in the equation. Taking $k = 2.0 \times 10^{-12}\mathrm{K}^{-3}/\mathrm{sec}$, find an approximate solution in this case and plot the result.

7. (*Harvesting*) One can modify the logistic population model to include harvesting (e.g., hunting) of animals. That is, assume that the animal population grows logistically while, at the same time, animals are being removed (by hunting, fishing, or whatever) at a constant rate of H animals per unit time. The model is

$$p' = rp\left(1 - \frac{p}{K}\right) - H.$$

   a) Choose new independent and dependent variables, $\tau = rt$ and $x = p/K$, and show that the model can be written in the form

   $$\frac{dx}{d\tau} = x(1 - x) - h,$$

   where h is a constant.

   b) Using the simplified model, determine the equilibria in the case $h > \frac{1}{4}$.

   c) Determine the stability of the equilibria.

   d) Explain how the population varies for different initial conditions. Does the population ever become extinct?

8. (*Life history*) In this exercise we introduce a simple model of growth of an individual organism over time. For simplicity, we assume it is shaped like a cube having sides equal to $L = L(t)$. Organisms grow because they assimilate nutrients and then use those nutrients in their energy budget for maintenance and to build structure. It is conjectured that the organism's

growth rate in volume equals the assimilation rate minus the rate food is used. Food is assimilated at a rate proportional to its surface area because food must ultimately pass across the cell walls; food is used at a rate proportional to its volume because ultimately cells are three-dimensional. Show that the differential equation governing its size $L(t)$ can be written

$$L'(t) = a - bL,$$

where a and b are positive parameters. What is the maximum length the organism can reach? Use separation of variables to show that if the length of the organism at time $t = 0$ is $L(0) = 0$ (it is very small), then the length is given by $L(t) = (a/b)(1 - e^{-bt})$. Does this function seem like a reasonable model for growth?

9. (*Pest outbreaks*) In a classical ecological study of budworm outbreaks in Canadian fir forests, researchers proposed that the budworm population N was governed by the law

$$N' = rN\left(1 - \frac{N}{K}\right) - P(N),$$

where the first term on the right represents logistics growth, and where $P(N)$ is a bird-predation rate given by

$$P(N) = \frac{aN^2}{N^2 + b^2}.$$

a) Sketch a generic plot of the bird-predation rate $P(N)$ versus N.

b) What are the dimensions of all the constants and variables in the model?

c) Define new dimensionless independent and dependent variables by

$$\tau = \frac{t}{b/a}, \quad n = \frac{N}{b},$$

and reformulate the differential equation in terms of those variables and certain dimensionless constants. (For this problem, a dimensionless form is extremely tractable compared to the dimensioned model.)

d) Working with the dimensionless equation, show that there is at least one and at most three positive equilibrium populations. What can be said about their stability? Hint: Write the equation in the form $dn/d\tau = nF(n)$ and plot the two terms of $F(n)$ to find their intersection points.

10. (*Epidemics*) In a fixed population of N individuals let $I = I(t)$ be the number of individuals infected by a certain disease and let $S = S(t)$ be the number susceptible to the disease with $I(t) + S(t) = N$. Assume that the rate that individuals are becoming infected is proportional to the number of infectives times the number of susceptibles, or $I' = aSI$, where the positive constant a is the disease transmission coefficient. Assume no individual gets over the disease once it is contracted. If $I(0) = I_0$ is a small number of individuals infected at $t = 0$, formulate an initial value problem for the number infected $I(t)$ at time t. Explain how the disease evolves. Over a long time, how many individuals in the population contract the disease? This type of disease, where no one recovers, is called an SI model.

11. (*Epidemics*) With the same notation as in the previous problem, suppose that infected individuals recover from the illness at the per capita rate r and then become susceptible again. This is an SIS model. Argue that the governing equations are

$$S' = -aSI + rI, \quad I' = aSI - rI,$$

where $I + S = N$, and formulate a differential equation for the number of infectives $I(t)$. Explain how the disease evolves from a small number of infectives and determine how many have the disease after a long time.

12. (*Dynamics*) The dynamical equation $x' = f(x)$ is said to have a potential function $V(x)$ if $V'(x) = -f(x)$. Show that x^* is an equilibrium for the equation if, and only if, $V'(x^*) = 0$. On any solution $x = x(t)$ of $x' = f(x)$, show that $V(|x(t)|)$ is strictly decreasing in time.

13. (*Economics*) In a model of price adjustment, the price per item $P = P(t)$ of a commodity is proportional to the difference between the demand D for the quantity and the quantity S that is supplied, or

$$\frac{dP}{dt} = \alpha(D - S).$$

If the supply is proportional to the price and the demand is inversely proportional to the price, set up a DE model for the price P and explain the reasoning behind the assumptions. Investigate the dynamics of the system, and find a formula for $P(t)$ if $P(0) = p_0$.

14. A function $V = V(x)$ is called a *potential energy function* for the autonomous equation $x' = f(x)$ if $f(x) = -V'(x)$. **(a)** Show that if $x = x(t)$ is any nonconstant solution to $x' = f(x)$, then $V(x(t))$ is a strictly decreasing function of t. **(b)** Show that if x^* is a critical point of $x' = f(x)$ and $V(x)$ has a local minimum at x^*, then x^* is stable.

1.5.2 Bifurcation

Differential equations almost always contain one or more parameters. It is of great interest to determine how equilibrium solutions depend upon those parameters and how the stability of an equilibrium behaves as the parameters change. It is observed that as the parameter slowly changes, there can be abrupt changes in stability of a steady state; stability can be lost or gained, and even the single equilibrium can split into two equilibria. A simple thought experiment can illustrate the phenomenon. Imagine a thin metal vertical rod standing upright on a table and a constant force F pushing downward on the top end of the rod. If F is small, nothing happens—the rod remains in its stable, vertical equilibrium. But as F is increased slowly, there is a value F^* when the rod buckles, moving it to a entirely different orientation. This phenomenon is called *bifurcation*, which means *dividing* or *splitting*.

Consider a quantitative example. The logistic equation

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right)$$

has two parameters: the growth rate r and the carrying capacity K. Let us add *harvesting*; that is, we remove animals at a constant rate $H > 0$. We can think of a fish population where fish are caught at a given rate H. Then we have the model

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right) - H. \tag{1.24}$$

We now ask how equilibrium solutions and their stability depend upon the rate of harvesting H.

There are three parameters in the problem, but we can simplify it using the new independent variables τ and x defined by

$$x = \frac{p}{K}, \quad \tau = rt.$$

This amounts to measuring the population relative to the carrying capacity and time relative to the inverse growth rate. In terms of these new variables, (1.24) simplifies to (check this!)

$$\frac{dx}{d\tau} = x(1-x) - h,$$

where $h = H/rK$ is a *single* parameter representing the ratio of the harvesting rate to the product of the growth rate and carrying capacity. We can now study the effects of changing h, a reduced harvesting parameter, to see how harvesting influences the steady-state fish populations in the model.

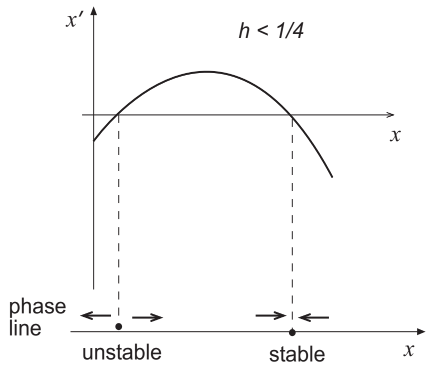

Figure 1.19 Plot of the phase line diagram $f(x) = x(1-x) - h$ for a fixed h, $0 < h \leq \frac{1}{4}$.

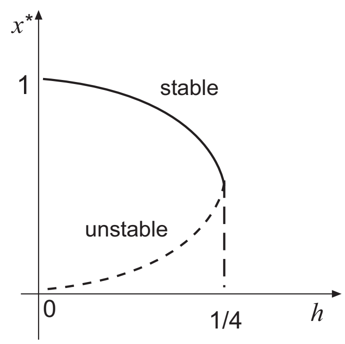

Figure 1.20 Bifurcation diagram: plot of the equilibria as a function of the bifurcation parameter h, $x^* = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4h}$. For $h > \frac{1}{4}$ there are no equilibria and for $h < \frac{1}{4}$ there are two, with the larger one being stable and the smaller one unstable (dashed). A bifurcation occurs at $h = \frac{1}{4}$.

The equilibrium solutions depend on h and are roots of the quadratic equation

$$f(x) = x(1-x) - h = 0, \qquad h > 0,$$

which are given by

$$x^* = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4h}. \tag{1.25}$$

If $h < \frac{1}{4}$ there are two positive equilibria. The graph of $f(x)$ in this case is concave down and the phase line shows that the smaller one is unstable, and the larger one is stable. (See Figure 1.19.) Notice, however, as h increases these two equilibria begin to come together, and at $h = \frac{1}{4}$ there is only a single unstable equilibrium at $x^* = \frac{1}{2}$. For $h > \frac{1}{4}$ the equilibrium populations cease to exist. In summary, when harvesting is small, there are two equilibria, one being stable; as harvesting increases the equilibrium disappears. We say that a bifurcation occurs at the value $h = \frac{1}{4}$. This is the value where there is a significant change in the character of the equilibria. For $h \geq \frac{1}{4}$ the population becomes extinct, regardless of the initial condition because $f(x) < 0$ for all x. All these facts can be conveniently represented on a **bifurcation diagram**. See Figure 1.20. *On a bifurcation diagram we plot the equilibrium solutions x^* versus the parameter h* as given in (1.25). In this context, h is called the **bifurcation parameter**. A plot of (1.25) is a parabola opening to the left. We observe that the upper branch of the parabola corresponds to the larger equilibrium (with the plus sign), and all solutions represented by that branch are stable; the lower branch, corresponding to the smaller solution (with the minus sign), is unstable.

In summary, if harvesting is small there is a stable and an unstable equilibrium; Nature usually selects out the stable one. As harvesting slowly increases, the two equilibria coalesce and at a critical value of h, the bifurcation point, the fish population dies out due to excessive removal by harvesting.

Example 1.35

Consider the simple equation

$$x' = x(h - x^2),$$

where h is a real parameter. The equilibria are $x^* = 0$ for all h, and $x^* = \pm\sqrt{h}$ when $h > 0$. The bifurcation diagram, plotting the equilibria x^* as functions of h, is shown in Figure 1.21. Notice that for $h > 0$ there are three equilibria, and for $h < 0$ there is just one. Therefore, if h is slowly increased from a negative number, at $h = 0$ a bifurcation occurs and the single equilibrium splits into three equilibria; the $x^* = 0$ equilibrium changes stability at the bifurcation point and two stable equilibria appear.

We can apply Theorem 1.32 to check stability of the different branches of the bifurcation diagram. We compute $f'(x)$, which will of course depend on h:

$$f'(x) = h - 3x^2.$$

Observe that

$$f'(0) = h.$$

Thus, $x^* = 0$ is stable if $h < 0$ and unstable if $h > 0$. In other words, the equilibrium $x^* = 0$ changes stability at $h = 1$. Next,

$$f'(\pm\sqrt{h}) = h - 3h = -2h.$$

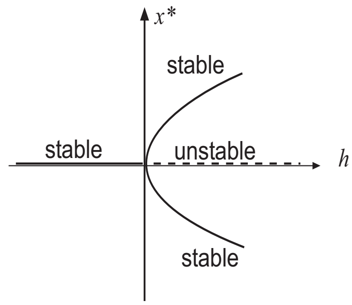

Figure 1.21 Bifurcation diagram: plots of x^* versus h. For obvious reasons, this is called a *pitchfork* bifurcation.

For positive h both branches, $x^* = \pm\sqrt{h}$, are stable. As an exercise, the reader can sketch the phase line diagram for $h > 0$ and $h < 0$ and observe that at $h = 0$ there is a dramatic change in the diagram. □

Notation. When we use prime to denote the derivative, we have to be careful to understand what the prime means. For example, $f'(x)$ means the derivative of f with respect to x, whereas a prime on x means a time derivative because $x = x(t)$ is a function of time. We almost always know from context about what derivative we are taking. If there is confusion, we write out the derivative more specifically, such as df/dx or dx/dt.

EXERCISES

1. The following models contain a parameter h. For each model: (a) find the equilibria in terms of h and determine their stability using Theorem 1.32, insofar as possible. (b) Construct a bifurcation diagram showing how the equilibria depend upon h; label the branches of the curves in the diagram as unstable or stable.

   a) $x' = h - x^3$.
   b) $x' = (x-1)(x-h)$.
   c) $y' = hy - y^2$.
   d) $x' = (1-x)(x^2-h)$.
   e) $x' = (x-\sqrt{h})(3-hx)$.
   f) $y' = -(1+y)(y^2-h^2)$.

2. Consider the differential equation

$$\frac{dx}{dt} = \frac{x}{x^2+1}.$$

Use the analytic criterion in Theorem 1.32 to investigate the stability of $x = 0$.

3. Consider the model $y' = (\lambda - 1)y - y^3$, where λ is a parameter.

    a) Explain what is meant by the statement "As λ increases through 1, the stable solution $y = 0$ bifurcates into 2 stable solutions."

    b) Sketch a bifurcation diagram showing how the equilibria vary with λ. Label each branch of the curves shown in the bifurcation diagram as stable or unstable.

4. The biomass P of a plant grows logistically with intrinsic growth rate r and carrying capacity K. At the same time it is consumed by herbivores at a rate

$$\frac{aP}{b+P},$$

per herbivore, where a and b are positive constants. The model is

$$P' = rP\left(1 - \frac{P}{K}\right) - \frac{aPH}{b+P},$$

where H is the biomass of herbivores. Assume $aH > br$, and assume r, K, a, and b are fixed. Plot, as a function of P, the growth rate (first term) and the consumption rate (second term) for several values of H on the same set of axes, and identify the values of P that give equilibria. What happens to the equilibria as the herbivory H is steadily increased from a small value to a large value? Draw a bifurcation diagram showing this effect. That is, plot equilibrium solutions versus the parameter H. If the herbivory is slowly increased so that the plants become extinct, and then it is decreased slowly back to a low level, do the plants return?

5. A deer population grows logistically and is harvested at a rate proportional to its population size. The dynamics of population growth is modeled by

$$P' = rP\left(1 - \frac{P}{K}\right) - hP,$$

where $h > 0$ is the harvesting rate. Use a bifurcation diagram to explain the effects on the equilibrium deer population when h is slowly increased from a small value to a large value.

6. Draw a bifurcation diagram for the model $x' = x^3 - x + h$, where h is a parameter. Label branches of the curves as stable or unstable. (Such a model is called a *hysteresis* model.) Hint: Graph h versus u and rotate the plot.

7. Consider the model $u' = u(u - e^{\lambda u})$, where λ is a parameter. Draw the bifurcation diagram, plotting the equilibrium solution(s) u^* versus λ. Label each curve on the diagram as stable or unstable. Hint: Graph λ versus u.

8. Consider the differential equation $x' = ax^2 - 1,\ -\infty < a < +\infty$, where a is a parameter. Draw the bifurcation diagram and indicate stability of the various branches.

9. Consider the differential equation $y' = b - e^{-y^2}$, where b is a positive parameter. Draw the bifurcation diagram and indicate the stability of the equilibrium.

10. The phosphorus concentration $P = P(\tau)$ at time τ in a small lake is modeled by

$$\frac{dP}{d\tau} = I - sP + r\frac{P^n}{M^n + P^n}.$$

Here, I is the input rate, often caused by runoff and leaching of fertilizer from local farms, s is the rate of sedimentation and outflow, and the last term is the rate of recycling of phosphorus from the sediment; M and n are positive constants, and n is generally large. If P is large, the lake becomes putrid with algae overgrowth, a condition called *eutrophication*. It is observed that onset of eutrophication occurs quickly as the inflow increases.

a) Introduce new, dimensionless, independent and dependent variables $t = s\tau,\ p = P/M$ and derive the simplified model

$$\frac{dp}{dt} = a - p + \rho\frac{p^n}{1 + p^n}, \tag{1.26}$$

where a and ρ are appropriately defined constants.

b) Take $n = 8$ and $\rho = 5$. Graphically determine the equilibrium concentrations as the inflow a varies from 0.25 to 0.75, and explain the rapid onset of eutrophication. Hint: Plot $p - a$ and $\rho p^n/(1 + p^n)$ versus p on the same set of axes and observe how the number of equilibria, or intersections, change as a increases, or the line $p - a$ moves downward.

c) Draw the bifurcation diagram, plotting the equilibria versus a. Label the stability of the branches.

11. (Thermistors) A *thermistor* is a resistor that depends on temperature; as it gets hotter, the resistance increases. They are used in fuses and electrical appliances to prevent overheating. If a constant voltage V is applied across a thermistor and its temperature T depends only on time, then the energy

balance applied to the thermistor leads to the model

$$mcT' = \frac{V^2}{R(T)} - h(T - T_e),$$

where m is its mass, c is the specific heat, h is the heat loss coefficient, T_e is the environmental temperature, and $R(T)$ is the resistance.

a) Derive the model equation. Hint: Use Newton's law of cooling and the ohmic heating.

b) If $R(T)$ is a positive increasing function, show that there is a unique equilibrium temperature.

1.5.3 Existence of Solutions

We defined an **initial value problem** for a first-order differential equation as the problem

$$\text{(IVP)} \qquad \begin{cases} x' = f(t, x), \\ x(t_0) = x_0. \end{cases} \tag{1.27}$$

Geometrically, solving an initial value problem means determining a solution that passes through a specified point (t_0, x_0) in the tx plane.

There are many interesting mathematical questions about initial value problems.

1. **(Existence)** Given an initial value problem, is there always be a solution? This is the question of existence. Note that there may be a solution even if we cannot find a formula for it.
2. **(Uniqueness)** If there is a solution, is it the only solution? This is the question of uniqueness.
3. **(Interval of existence)** If there is a solution, for which times t does the solution to the initial value problem exist?

Resolution of these theoretical issues is an interesting and worthwhile endeavor, and it is the subject of advanced courses and books on differential equations. In this text we briefly discuss the matters. The next three examples illustrate why these are reasonable questions.

Example 1.36

The initial value problem

$$x' = x\sqrt{t-3}, \quad x(1) = 2,$$

has no solution because the derivative of x is not defined in an interval containing the initial time $t = 1$. There cannot be a solution curve passing through the point $(1, 2)$. □

Example 1.37

Consider the initial value problem

$$x' = 2\sqrt{x}, \quad x(0) = 0.$$

Both $x(t) = 0$ and $x(t) = t^2$ are solutions to this problem on the interval $-\infty < t < \infty$. Thus it does not have a unique solution. □

The following theorem, proved in advanced books, provides partial answers to the questions raised above. The theorem basically states that if the right side $f(t, x)$ of the differential equation is nice enough in a domain in the tx plane containing the point (t_0, x_0), then there is a unique solution that passes through the that point. A. Cauchy, in the 1820's, was the first to prove such an existence theorem.

Theorem 1.38

Assume the function $f(t, x)$ and its partial derivative[3] $f_x(t, x)$ are continuous in a rectangle $a < t < b$, $c < x < d$. Then, for any value t_0 in $a < t < b$ and x_0 in $c < x < d$, the initial value problem

$$\begin{cases} x' = f(t, x), \\ x(t_0) = x_0, \end{cases} \tag{1.28}$$

has a unique solution valid on some open interval $a < \alpha < t < \beta < b$ containing t_0. □

The theorem tells us nothing about how big the interval (α, β) is. The **interval of existence** is the set of all time values for which the solution to the initial value problem exists. That interval may not extend out to the boundaries of the rectangle. Theorem 1.38 is called a *local* existence theorem because it guarantees a solution only in a neighborhood of the initial time t_0. Notice that t_0 and x_0 have to lie in open intervals and not on the boundary of those intervals.

[3] We use subscripts to denote partial derivatives, and so $f_x = \partial f / \partial x$.

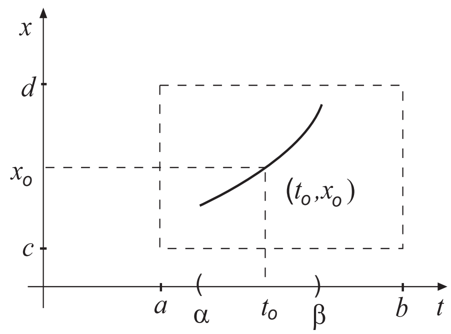

Figure 1.22 Solution to an initial value problem. The fundamental questions are: (a) is there a solution curve passing through the given point, (b) is the curve the only one, and (c) what is the interval (α, β) on which the solution exists.

Example 1.39

For the differential equation

$$x' = 1 + x^2.$$

$f(t, x) = 1 + x^2$, and $f_x(t, x) = 2x$. Clearly both are continuous in the entire tx plane. If we choose any initial condition $x(t_0) = x_0$ we are guaranteed to find a rectangle about it where a unique solution exists. Say we choose $x(0) = 0$. Then the unique solution is $x(t) = \tan x$ which exists in the interval $(-\pi/2, \pi/2)$. So, no matter how large we choose the rectangle, the solution will not extend beyond this interval. Hence, even though f and f_x are very nice in a large domain, the solution may of the IVP may be valid in only a small interval. □

Example 1.40

In Example 1.37 what went wrong? Here, $f(t, x) = 2\sqrt{x}$, and its partial derivative $f_x(t, x) = 1/\sqrt{x}$, are both continuous in the upper half-plane $x > 0$. However, the initial point $t = 0$, $x = 0$ lies on the *boundary* of that region. There is no open rectangle containing the initial point where f and f_x are continuous. The theorem does not apply. However, if the initial value is entirely in the region $x > 0$ there is a unique solution. □

Example 1.41

(**Linear equations**) First-order linear equations have very nice properties.

Consider the IVP

$$\text{(IVP)} \qquad \begin{cases} x' + p(t)x = q(t), \\ x(t_0) = x_0. \end{cases} \tag{1.29}$$

If p and q are continuous on any open interval I containing t_0, then there is a unique solution to the IVP on the entire interval I. The proof of this result follows from the fact that linear equations have analytic solutions containing the integrals of continuous functions, which are in fact differentiable. Take the initial value problem

$$tx' + 2x = 4t^2, \quad x(1) = 2.$$

In normal form

$$x' + \frac{2}{t}x = 4t.$$

Here, $p(t) = 2/t$ and $q(t) = 4t$. The function $p(t)$ is continuous on $t < 0$ and $t > 0$, but not at $t = 0$; and $q(t)$ is continuous for all t. The initial time $t = 1$ lies in the interval $t > 0$. So there is a unique solution on $0 < t < +\infty$. In fact, the solution is

$$x(t) = t^2 + \frac{1}{t^2}, \quad t > 0.$$

For linear equations we can usually find the interval of existence by simple inspection. □

EXERCISES

1. State explicitly how you know that the IVP

$$x' = (t^2 + 1)x - t, \quad x(1) = 3$$

has a unique solution valid in some interval containing $t = 1$.

2. Can you guarantee that the IVP

$$x' = \frac{tx(1-x)}{1-t^2}, \quad x(0) = \frac{1}{2}$$

has a unique solution valid in some interval containing $t = 0$?

3. For which initial points (t_0, x_0) are you assured that the initial value problem

$$\frac{dx}{dt} = \ln(t^2 + x^2), \quad x(t_0) = x_0$$

has a unique solution?

4. Find the largest interval where the solution to the IVP exists:

a) $t(t-5)x' + x = e^{-t}, \ x(2) = 1.$ *b*) $x' + \frac{1}{t-3}x = \frac{1}{t-7}, \ x(4) = 1.$

5. Find the regions in the xt plane where the hypotheses of Theorem 1.38 hold:

$$a) \quad x' = \frac{2+t^2}{3x-x^3}. \qquad\qquad b) \quad (2t+5x)x' = t-x.$$

6. In each of the following problems, find how the solution depends on the initial condition $x(0) = x_0$:

$$a) \quad x' = -\frac{4t}{x}. \qquad b) \quad x' + x^3 = 0. \qquad (c) \quad x' = x^2.$$

7. Verify that the initial value problem $x' = \sqrt{x}$, $x(0) = 0$, has infinitely many solutions of the form

$$x(t) = \begin{cases} 0, & t \leq a \\ \frac{1}{4}(t-a)^2, & t > a, \end{cases}$$

where $a > 0$ is fixed. Sketch these solutions for three different values of a. Why might you not be surprised at this result?

8. Verify that the linear initial value problem

$$x' = \frac{2(x-1)}{t}, \quad x(0) = 1,$$

has a continuously differentiable solution (i.e., a solution whose first derivative is continuous) given by

$$x(t) = \begin{cases} at^2 + 1, & t < 0, \\ bt^2 + 1, & t > 0, \end{cases}$$

for any constants a and b. Yet, there is no solution if $x(0) \neq 1$. Do these facts contradict Theorem 1.38?

2
Second-Order Linear Equations

In this chapter we study second-order linear differential equations of the form

$$ax'' + bx' + cx = f(t)$$

and their applications to classical mechanics and electrical circuits. These applications are standard fare and a centerpiece in both elementary physics and engineering courses, and they serve as prototypes for oscillating systems, oscillating systems with dissipation, or damping, and forced vibrations that occur in all areas of pure and applied science. In the final sections of the chapter we extend the coverage to linear equations with variable coefficients.

2.1 Classical Mechanics

Newton's second law is familiar from high school physics and beginning calculus courses. It is the fundamental law of classical particle dynamics and is perhaps the most well known law in elementary physics. Its simple statement is: *force equals mass times acceleration*, or $F = ma$. When an external force acts on a particle of mass m, it changes the momentum, or inertia, in the system. If $x = x(t)$ denotes the position of the particle, then the particle undergoes an acceleration given by the second derivative, $x''(t)$. Thus Newton's law takes the form $mx'' = F$, which is a second-order differential equation for the position x. In the general case, the external force could depend on time t, the position

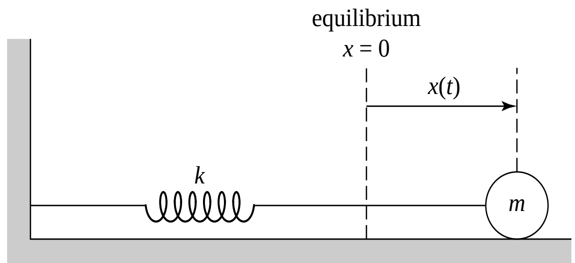

Figure 2.1 Spring–mass oscillator. $x(t)$ is the displacement of the mass at time t, where $x = 0$ is the equilibrium positon; $x(t) > 0$ when the mass is displaced to the right.

x, and the velocity x'. Therefore Newton's law of motion can be expressed generally as

$$mx'' = F(t, x, x'), \qquad (\textbf{Newton's Second Law})$$

where the form of the force $F(t, x, x')$ is prescribed.

In dynamics we expect to impose *two* initial conditions,

$$x(0) = x_0, \quad x'(0) = x_1,$$

where x_0 is the initial position and x_1 is the initial velocity. In general terms, the program of classical mechanics is *deterministic*; that is, if the initial state (position and velocity) of a system is known, as well as the forces acting on the system, then the state of the system is determined for all times $t > 0$. Practically, this means we solve the initial value problem above associated with Newton's law to determine the evolution of the system $x = x(t)$.

2.1.1 Oscillations and Dissipation

Oscillatory behavior is a common phenomenon in mechanical, biological, electrical, atomic, and other physical systems. We begin with a prototype of a simple oscillatory system, a mass connected to a spring.

Example 2.1

(**Oscillator**) Imagine a mass m lying on a table and connected to a spring, which is in turn attached to a rigid wall (Figure 2.1). At time $t = 0$ we displace the mass a positive distance x_0 to the right of equilibrium and then release it. If we ignore friction on the table then the mass executes *simple harmonic motion*; that is, it slides back and forth at a fixed frequency and amplitude. Following the doctrine of mechanics we write down Newton's second law of motion, $mx'' = F_s$, where the state function $x = x(t)$ is the position of the mass at time t; we take $x = 0$ to be the equilibrium position and $x > 0$ to the right; the spring exerts external force F_s, which must be prescribed. Experiments confirm that if the displacement is not too large (which we assume), then the force exerted by the spring on the mass is proportional to its displacement from equilibrium, or

$$F_s = -kx. \qquad \text{(Hooke's Law)} \tag{2.1}$$

The minus sign appears because the force opposes positive motion, which is to the right. The force is negative when $x > 0$, and positive if $x < 0$. The proportionality constant $k > 0$ (having dimensions of force per unit distance) is called the **spring constant**, or **stiffness** of the spring, and Equation (2.1) is called **Hooke's law**. Not every spring behaves in this manner, but Hooke's law is used as a model for some springs. In engineering such a law is called a **constitutive relation**; it is an empirical result rather than a law of nature. This is an example of a linear force that depends only on the position x.

To give more justification for Hooke's law, suppose the force F_s depends on the displacement x through $F = F_s(x)$, with $F_s(0) = 0$. By Taylor's theorem,

$$\begin{aligned} F_s(x) &= F_s(0) + F_s'(0)x + \frac{1}{2}F_s''(0)x^2 + \cdots \\ &= -kx + \frac{1}{2}F_s''(0)x^2 + \cdots, \end{aligned}$$

where we *defined* $F_s'(0) = -k$. So Hooke's law is a good approximation if the displacement is small, allowing the higher-order terms in the series to be neglected.

We can measure the stiffness k of a spring by attaching it to the ceiling without the mass. Then we attach the mass m and measure the elongation L after it comes to rest. The force of gravity mg (downward) must balance the restoring force kx (upward) of the spring, so $mg = kL$. Therefore,

$$k = \frac{mg}{L}.$$

Newton's law, or the equation of motion of the system, is therefore

$$mx'' = -kx. \tag{2.2}$$

This is the **spring-mass equation**. The initial conditions are $x(0) = x_0$, the position where the mass is released, and the velocity $x'(0) = x_1$ given to it at time $t = 0$.

As a special case, suppose the initial velocity is zero. That is, we just displace the mass to x_0 and release it. The initial conditions are

$$x(0) = x_0, \quad x'(0) = 0.$$

We expect oscillatory motion. Assuming a solution of (2.2) of the form $x(t) = A\cos\omega t$ for some unknown frequency ω and amplitude A, we find upon substitution into (2.2) that $\omega = \sqrt{k/m}$ and $A = x_0$. (Verify this!) Therefore, the position of the mass at time t is given by

$$x(t) = x_0 \cos\sqrt{k/m}\, t.$$

This solution is an oscillation of amplitude x_0, natural frequency $\sqrt{k/m}$, and period 2π divided by the frequency, or $2\pi\sqrt{m/k}$. □

Now we add an additional force, that of the frictional force of the table exerted on the mass. Friction is a force that opposes positive motion and it **dissipates**, or decreases, the energy in the system. Dissipation in mechanical systems include friction, air resistance, and so on, which are also called damping forces. In circuits, electrical resistance dissipates the energy in the circuit. These energies often are transformed to heat energy.

Example 2.2

(**Damped Oscillator**) Assume there is friction as the mass slides on the table. The simplest constitutive relation is to take the frictional force to be proportional to how fast the mass is moving, or its velocity x'. Thus

$$F_d = -\gamma x', \qquad \text{(damping force)}$$

where $\gamma > 0$ is the **damping coefficient** (mass per time). If the mass is moving to the right, or $x' > 0$, the damping retards the motion and $F_d < 0$. Therefore, the total external force is the sum

$$F = F_s + F_d = -kx - \gamma x'.$$

The equation of motion is

$$mx'' = -\gamma x' - kx. \qquad \text{(damped oscillator)}$$

This equation is called the **damped oscillator equation**. Both forces have negative signs because each opposes positive (to the right) motion. For this case we expect an oscillatory solution with a decreasing amplitude during each oscillation because of the presence of friction. One such a solution, a damped oscillation, takes the form of

$$x(t) = Ae^{-\lambda t}\cos\omega t,$$

where A is some amplitude, λ is a decay rate, and ω is the frequency. □

The Mechanical-Electrical Analogy

There is great similarity between the damped spring-mass system and an RCL circuit. We can write the damped oscillator equation with unknown displacement $x(t)$ as

$$mx'' + \gamma x' + kx = 0. \qquad (\textbf{damped oscillator})$$

Interestingly enough, from Section 1.5, the current $I(t)$ in an RCL circuit with no electromotive force (emf) satisfies

$$LI'' + RI' + \frac{1}{C}I = 0, \qquad (\textbf{RCL circuit})$$

which has *exactly* the same form. This similarity is a classical example of the unifying nature of mathematics in science. The similarity between these two models is called the **mechanical–electrical analogy**:

- The spring constant k is analogous to the inverse capacitance $1/C$; both a spring and a capacitor *store energy* in the system.
- The damping constant γ is analogous to the resistance R; both friction in a mechanical system and a resistance in an electrical system *dissipate energy*, often in the form of heat.
- The mass m is analogous to the inductance L; both represent *inertia* in the system; a large mass or inductance causes the velocity or current, respectively, to resist change. In the case of a circuit, an inductor (or coil) stores energy in its magnetic field which resists changes in current.

In every equation we encounter, we want to understand the meaning of each term. In the analogy above the first term is the inertia term, the second term is the dissipation or energy loss term, and the third is an energy storage term. Many of the equations we examine in this chapter can be regarded as either circuit equations or mechanical equations. Common among them are the physical properties of inertia, damping, and oscillation. In Section 2.3 we consider additional forces on the system due to external forcing or an electromotive force.

Energy Considerations

To get a better idea about the role of dissipation in a system, let us think a little deeper about energy. For a damped spring-mass system, the governing equation is

$$mx'' + \gamma x' + kx = 0.$$

To get energy expressions, multiply this equation by the velocity x' to obtain

$$mx'x'' + \gamma x'x' + kxx' = 0.$$

Each term has units of energy per time. By the chain rule

$$\frac{d}{dt}x'(t)^2 = 2x'(t)x''(t), \quad \text{and} \quad \frac{d}{dt}x(t)^2 = 2x(t)x'(t).$$

Therefore, the energy equation can be written

$$\frac{d}{dt}\left[\frac{1}{2}m(x')^2 + \frac{1}{2}kx^2\right] = -\gamma x'x'. \tag{2.3}$$

The two terms inside the left bracket are the kinetic energy and potential energy in the system:

$$T = \frac{1}{2}m(x')^2 \quad (\textbf{kinetic energy}); \quad V = \frac{1}{2}kx^2 \quad (\textbf{potential energy})$$

To understand potential energy, recall that the *force is the negative derivative of the potential*. The force is $F(x) = -kx$, so the potential energy due to that force is the negative integral of the force, or

$$V(x) = -\int(-kx)dx = \frac{1}{2}kx^2.$$

Therefore (2.3) is an energy dissipation law

$$\frac{dE}{dt} = \frac{d}{dt}[T + V] = -\gamma x'x',$$

stating that the *total* energy $E = T + V$ in the system is dissipated at the rate $-\gamma x'x'$. Energy per time is *power*, so (2.3) is the power lost in the system.

If there is no damping in the system then $\gamma = 0$ and we have

$$\frac{1}{2}m(x')^2 + \frac{1}{2}kx^2 = E, \quad E \text{ constant.}$$

This is the **conservation of energy** law.

EXERCISES

1. When a mass of 0.3 kg is placed on a spring hanging at rest from the ceiling, it elongates the spring 5 cm. What is the stiffness k of the spring?

2. **(a)** Beginning with the RCL circuit equation expressed in terms of charge Q on the capacitor, or

$$LQ'' + RQ' + \frac{1}{C}Q = 0,$$

derive the energy dissipation law corresponding to a spring-mass oscillator (2.3). **(b)** Identify the energy in the inductor and on the capacitor. **(c)** Show that the power lost by the resistor is $-RI^2$. **(d)** If there is no resistor, what is the conservation of energy law for the circuit?

3. We derived the spring-mass equation for a mass moving horizontally on a table top. This exercise shows that the same equation holds for a mass oscillating on a vertical, perfectly elastic string (or a spring) under the influence of the force of gravity mg. We assume Hooke's law holds for the string, that is, the force exerted by the string is proportional to its displacement. See the set up in Figure 2.2, where we introduce two coordinate systems, x and y, to measure displacement. First, only the string of natural length L is attached with no mass. This is position $y = 0$. Then we attach the mass, and at equilibrium it reaches a natural length $L + \Delta L$; this is position $x = 0$. Next we pull the mass down a positive distance and release it, and it undergoes oscillatory motion.

   a) Use Newton's second law and Hooke's law to justify the equation of motion

   $$my'' = -ky + mg.$$

   b) In the x coordinate system show that the equation of motion is

   $$mx'' = -kx,$$

   where the gravitational force does not appear. Hint: Note that $y = x + \Delta L$ and use the definition of the stiffness, $k\Delta L = mg$.

   c) Show that if damping occurs, then the governing equation is $mx'' = -kx - \gamma x$.

2.2 Equations with Constant Coefficients

Both the damped spring-mass equation and the RCL circuit equation have the form, namely,

$$ax'' + bx' + cx = 0, \tag{2.4}$$

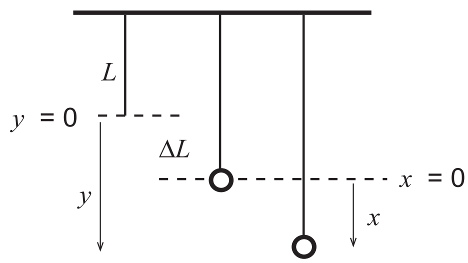

Figure 2.2 Two coordinate systems x and y for the motion of a mass m attached to a perfectly elastic string, or a spring, subject to both a gravitational force mg downward and a restoring force given by Hooke's law.

where a, b, and c are constants. An equation of the form (2.4) is called a **homogeneous linear equation with constant coefficients**. The word **homogeneous** refers to the fact that the right side is zero, meaning there is no external forces acting on the mass or no emf in the circuit. (In Section 2.3 we include these types of forces.) Equation (2.4) can be accompanied by initial conditions of the form

$$x(0) = x_0, \quad x'(0) = x_1. \tag{2.5}$$

Here, we are using $x = x(t)$ as the generic state function.

The problem of solving (2.4) subject to (2.5) is called the **initial value problem** (IVP). In (2.5) the initial conditions are given at $t = 0$, which is the common case, but they could be given at any time $t = t_0$. Finally, in the spring-mass problem and RCL circuit problem the constants a, b, and c are nonnegative, but our analysis is valid for any values of the constants.

2.2.1 The General Solution

We develop a simple technique to solve homogeneous linear equation

$$ax'' + bx' + cx = 0.$$

Fundamental to the discussion is the following existence–uniqueness theorem, which is proved in advanced texts; it also includes a definitive statement about the interval where solutions are valid.

Theorem 2.3

(**Existence-Uniqueness**) The initial value problem (2.4)–(2.5) has a unique solution that exists on $-\infty < t < \infty$. □

The issue is how to find the solution. For the constant coefficient equation (2.4), with no initial conditions, we demonstrate that there are always exactly two **independent solutions**, say $x_1(t)$ and $x_2(t)$, meaning one is not a constant multiple of the other; they are not proportional. Such a set of solutions, $x_1(t)$, $x_2(t)$, is called a basic, or **fundamental set**. Further, if we multiply each by an arbitrary constant and form the linear combination

$$x(t) = c_1 x_1(t) + c_2 x_2(t), \tag{2.6}$$

where c_1 and c_2 are arbitrary constants, then we can easily check that $x(t)$ is also a solution to the differential equation (2.4). (This is the superposition principle—see Exercise 3.) The linear combination (2.6) is called the **general solution** to (2.4), which means that *all* solutions to (2.4) are contained in this linear combination for different choices of the constants c_1 and c_2. In solving the initial value problem we use the initial conditions (2.5) to *uniquely* determine the constants c_1 and c_2 in (2.6).

Theorem 2.4

(**General Solution**) Let $x_1(t)$, $x_2(t)$ be a fundamental set of solutions of (2.4), and let $\phi(t)$ be any other solution. Then there exists unique values of the constants c_1 and c_2 such that

$$\phi(t) = c_1 x_1(t) + c_2 x_2(t). \quad \square$$

To prove this result, let $x_1(t)$ and $x_2(t)$ be the unique solutions that satisfy the initial conditions

$$x_1(0) = 1, \; x_1'(0) = 0, \qquad x_2(0) = 0, \; x_2'(0) = 1,$$

respectively, and let $\phi(t)$ be *any* solution of (2.4). $\phi(t)$ will satisfy some conditions at $t = 0$, say, $\phi(0) = A$ and $\phi'(0) = B$. But the function

$$x(t) = Ax_1(t) + Bx_2(t)$$

satisfies those same initial conditions, $x(0) = A$ and $x'(0) = B$. By the uniqueness theorem, Theorem 2.3, $\phi(t) = x(t)$ and so $c_1 = A$, $c_2 = B$. So the solution $\phi(t)$ is contained in the general solution. $\square$

Construction of Solutions

Our strategy now is to find two independent solutions $x_1(t)$ and $x_2(t)$ of (2.4). We suspect something of the form

$$x(t) = e^{\lambda t},$$

where λ is a constant to be determined, might work because every term in (2.4) has to be the same type of function for cancelation to occur; thus, x, x', and x'' must be the same form, which suggests an exponential function for x. Substitution of $x = e^{\lambda t}$ into (2.4) instantly leads to

$$a\lambda^2 + b\lambda + c = 0, \qquad \text{(characteristic equation)} \tag{2.7}$$

which is a quadratic equation for the unknown λ. Equation (2.7) is called the **characteristic equation**. Using the quadratic formula, we obtain its roots

$$\lambda = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right). \qquad \text{(eigenvalues)}$$

These roots of the characteristic equation are called the **eigenvalues** corresponding to the differential equation (2.4). Each value of λ gives a solution $x(t) = e^{\lambda t}$ to the equation

$$ax'' + bx' + cx = 0.$$

Clearly, the values of λ could be real numbers or complex numbers. Thus, there are three cases, depending upon whether the discriminant $b^2 - 4ac$ is positive, zero, or negative.

Case 1. If $b^2 - 4ac > 0$, then there are two real unequal eigenvalues λ_1 and λ_2. Hence, there are two independent, exponential-type solutions

$$x_1(t) = e^{\lambda_1 t}, \quad x_2(t) = e^{\lambda_2 t}, \qquad \lambda_1 \neq \lambda_2.$$

Therefore the general solution to (2.4) is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}. \tag{2.8}$$

Case 2. If $b^2 - 4ac = 0$ then there is a double root $\lambda_1 = -b/2a$, $\lambda_2 = -b/2a$. Then one solution is $x_1(t) = e^{\lambda t}$, where $\lambda = -b/2a$. A second independent solution in this case is $x_2(t) = te^{\lambda t}$. (Later we show why this solution occurs.) Therefore the general solution to (2.4) in this case is

$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}. \tag{2.9}$$

Case 3. If $b^2 - 4ac < 0$ then the roots, or eigenvalues, of the characteristic equation are complex conjugates having the form[1]

$$\lambda_1 = \alpha + i\beta, \qquad \lambda_2 = \alpha - i\beta,$$

[1] Notation: Any complex number z can be written $z = u + iv$, where u and v are real numbers; u is called *the real part of* z and v is called *the imaginary part of* z. Similarly, if $z(t) = u(t) + iv(t)$ is a complex function, then $u(t)$ and $v(t)$ are its real and imaginary parts, respectively. The numbers $u + iv$ and $u - iv$ are called *complex conjugates.*

where

$$\alpha = -\frac{b}{2a}, \quad \beta = \frac{1}{2a}\sqrt{4ac - b^2}.$$

Therefore two *complex-valued* solutions of (2.4) are

$$x_1(t) = e^{(\alpha + i\beta)t}, \quad x_2(t) = e^{(\alpha - i\beta)t}.$$

But we want *real-valued* solutions. This case requires a more detailed discussion, which is in Section 2.2.3.

2.2.2 Real Eigenvalues

We first present examples when the characteristic equation has real eigenvalues.

Example 2.5

(**Real, unequal eigenvalues**) The differential equation

$$x'' - x' - 12x = 0$$

has characteristic equation

$$\lambda^2 - \lambda - 12 = 0 \quad \text{or} \quad (\lambda + 3)(\lambda - 4) = 0.$$

The eigenvalues are

$$\lambda = -3, 4.$$

These are real and distinct and so a fundamental set of solutions is $x_1(t) = e^{-3t}$, $x_2(t) = e^{4t}$. The general solution to the equation is

$$x(t) = c_1 e^{-3t} + c_2 e^{4t}. \qquad \square$$

Example 2.6

(**Real, equal eigenvalues**) The differential equation

$$x'' + 4x' + 4x = 0$$

has characteristic equation

$$\lambda^2 + 4\lambda + 4 = 0 \quad \text{or} \quad (\lambda + 2)(\lambda + 2) = 0.$$

The eigenvalues are

$$\lambda = -2, -2,$$

which are real and equal (a double root). The fundamental set of solutions is $x_1(t) = e^{-2t}$, $x_2(t) = te^{-2t}$. The general solution is

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}. \qquad \square$$

Example 2.7

If the eigenvalues are $\lambda = 6$ and $\lambda = -2$, find the differential equation. The characteristic equation must factor into $(\lambda - 6)(\lambda + 2) = 0$. Multiplying out, the characteristic equation is then $\lambda^2 - 4\lambda - 12 = 0$. The differential equation must be $x'' - 4x' - 12 = 0$. □

EXERCISES

1. Find the general solution:

   (a) $x'' - 4x' + 4x = 0$.

   (b) $x'' - 2x' = 0$.

   (c) $\frac{1}{2}x'' + x' + \frac{1}{2}x = 0$.

   (d) $x'' + 4x' + 3x = 0$.

2. In Exercise 1, parts (a) through (d), find the solution satisfying the initial conditions $x(0) = 1$, $x'(0) = 0$.
3. In Exercise 1, part (d), find the solution satisfying the initial conditions $x(0) = -1, x'(0) = 2$.
4. Which of the equations in Exercise 1 can have a physical meaning in the context of a spring-mass or damped spring-mass oscillator?
5. Suppose $x_1(t)$ and $x_2(t)$ are two solutions to $ax'' + bx' + cx = 0$. Show that $x(t) = c_1x_1(t) + c_2x_1(t)$ is also a solution.
6. Find an equation that has general solution $x(t) = c_1e^{4t} + c_2e^{-6t}$.
7. Find an equation that has solution $x(t) = e^{-3t} + 2te^{-3t}$. What are the initial conditions?
8. Show that the graph of any solution of the equation $x'' + 2ax' + a^2x = 0$, $a > 0$, with two negative equal eigenvalues cannot cross the t axis more than once. Hint: Find the general solution and assume, by way of contradiction, that there are two crossings.

2.2.3 Complex Eigenvalues

We return to Case 3, the case of complex eigenvalues, or roots, of the characteristic equation. The following important theorem indicates that a complex-valued solution always defines a set of two real-valued solutions, or a fundamental set.

Theorem 2.8

If $x(t) = g(t) + ih(t)$ is a complex-valued solution of the differential equation (2.4),

$$ax'' + bx' + cx = 0,$$

then its real and imaginary parts, $x_1(t) = g(t)$ and $x_2(t) = h(t)$, are real-valued solutions. □

To see why this is true, substitute the solution $x = g(t) + ih(t)$ into the differential equation $ax'' + bx' + cx = 0$ to get

$$a(g(t) + ih(t))'' + b(g(t) + ih(t))' + c(g(t) + ih(t)) = 0,$$

and then rearrange to get

$$[ag''(t) + bg'(t) + cg(t)] + i[ah''(t) + bh'(t) + ch(t)] = 0.$$

The left side is complex-valued and equal to zero. The only way complex numbers can equal zero is if both the real and imaginary parts are zero,[2] or

$$ag''(t) + bg'(t) + cg(t) = 0, \quad ah''(t) + bh'(t) + ch(t) = 0.$$

But this means $g = g(t)$ and $h = h(t)$ are real-valued solutions to (2.4). □

Let us take the first of the complex solutions, $e^{(\alpha+i\beta)t}$, given in Case 3 and expand it into its real and imaginary parts using **Euler's formula:**

$$e^{i\beta t} = \cos\beta t + i\sin\beta t. \qquad \text{(Euler's formula)}$$

We have

$$\begin{aligned} e^{(\alpha+i\beta)t} &= e^{\alpha t}e^{i\beta t} = e^{\alpha t}(\cos\beta t + i\sin\beta t) \\ &= e^{\alpha t}\cos\beta t + ie^{\alpha t}\sin\beta t. \end{aligned}$$

Therefore, by Theorem 2.8,

$$x_1(t) = e^{\alpha t}\cos\beta t \quad \text{and} \quad x_2(t) = e^{\alpha t}\sin\beta t$$

are two real independent solutions to Equation (2.4). If we take the second of the complex solutions, $e^{(\alpha-i\beta)t}$, with the minus sign, then we get the same two independent, real solutions. Consequently, we can refine Case 3 and state:

[2] If $a + bi = 0$, where a and b are real, then, necessarily, $a = b = 0$.

Case 3. If the eigenvalues of the characteristic polynomial are complex, $\lambda = \alpha \pm i\beta$, then the general solution to (2.4) is

$$x(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t. \tag{2.10}$$

If $\alpha < 0$, these solutions represent **decaying oscillations** and if $\alpha > 0$, these solutions represent **growing oscillations**. □

Example 2.9

(**Complex eigenvalues**) The differential equation

$$x'' + 2x' + 5x = 0$$

has characteristic equation

$$\lambda^2 + 2\lambda + 5 = 0.$$

The quadratic formula gives complex roots

$$\lambda = -1 \pm 2i.$$

Therefore the general solution is

$$x(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t. \qquad \square$$

Purely imaginary eigenvalues

Purely imaginary eigenvalues, $\lambda = \pm i\beta$, are a special case of complex eigenvalues in Case 3. From the general solution (2.10) we set $\alpha = 0$ and get the general solution

$$x(t) = c_1 \cos \beta t + c_2 \sin \beta t.$$

These solutions are purely **oscillatory** with frequency β and period $2\pi/\beta$.

Example 2.10

Consider the initial value problem

$$x'' + 7x = 0,$$
$$x(0) = 1, \quad x'(0) = 2.$$

The characteristic polynomial is $\lambda^2 + 7 = 0$, or $\lambda^2 = -7$. Taking the square root we get two purely imaginary eigenvalues,

$$\lambda = \pm\sqrt{7}\, i.$$

Therefore the general solution to the differential equation is

$$x(t) = c_1 \cos\sqrt{7}\,t + c_2 \sin\sqrt{7}\,t.$$

Applying the initial conditions gives (noting $\sin 0 = 0,\ \cos 0 = 1$)

$$x(0) = c_1 = 1.$$

Then,

$$x'(t) = -\sqrt{7}\,c_1 \sin\sqrt{7}\,t + \sqrt{7}\,c_2 \cos\sqrt{7}\,t.$$

Hence,

$$x'(0) = \sqrt{7}\,c_2 = 2, \quad \text{or} \quad c_2 = \frac{2}{\sqrt{7}}.$$

Therefore the solution to the initial value problem is

$$x(t) = \cos\sqrt{7}\,t + \frac{2}{\sqrt{7}} \sin\sqrt{7}\,t.$$

Notice that the solution is purely oscillatory, or periodic of period $2\pi/\sqrt{7}$ and frequency $\sqrt{7}$. □

For easy reference, in the following table we summarize the form of solutions.

Table 2.1 Table showing the general solution of the equation $ax'' + bx' + cx = 0$. The eigenvalues λ are the two roots of the characteristic equation $a\lambda^2 + b\lambda + c = 0$.

eigenvalues	general solution
$\lambda_1 \neq \lambda_2$ (real, unequal)	$x(t) = e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$
$\lambda_1 = \lambda_2$ (real, equal)	$x(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$
$\lambda = \alpha \pm \beta i$ (complex)	$x(t) = c_1 e^{\alpha t} \cos\beta t + c_2 e^{\alpha t} \sin\beta t$
$\lambda = \pm\beta i$, (purely imaginary)	$x(t) = c_1 \cos\beta t + c_2 \sin\beta t$

Hyperbolic form. The equation

$$x'' - a^2 x = 0$$

occurs often and has real eigenvalues $\lambda = \pm a$; the general solution is

$$x(t) = c_1 e^{at} + c_2 e^{-at},$$

which is exponential. This equation can also be written in terms of the **hyperbolic functions** cosh and sinh as

$$x(t) = c_1 \cosh at + c_2 \sinh at,$$

where

$$\cosh at = \frac{e^{at} + e^{-at}}{2}, \quad \sinh at = \frac{e^{at} - e^{-at}}{2}.$$

It is sometimes it is easier to apply initial conditions with the hyperbolic form of the general solution. □

EXERCISES

1. Find the general solution:

   (a) $x'' + x' + 4x = 0$.

   (b) $x'' - 4x' + 6x = 0$.

   (c) $x'' + 9x = 0$.

   (d) $x'' - 12x = 0$.

   (e) $2x'' + 3x' + 3x = 0$.

   (f) $\frac{1}{2}x'' + \frac{5}{6}x' + \frac{2}{9}x = 0$.

2. In Exercise 1, parts (a) through (f), find the solution satisfying the initial conditions $x(0) = 1$, $x'(0) = 0$.
3. Find an equation that has solution $x(t) = \sin 4t + 3\cos 4t$. What are the initial conditions?
4. Find an equation having general solution $x(t) = c_1 \cosh 5t + c_2 \sinh 5t$, where c_1 and c_2 are arbitrary constants. Find the arbitrary constants when $x(0) = 2$ and $x'(0) = 0$.
5. Find an equation with solution $x(t) = e^{-2t}(\sin 4t + 3\cos 4t)$. What are the initial conditions?
6. Use Euler's formula to show $1 + e^{i\pi} = 0$. Why do you think this formula is interesting?
7. Find a differential equation that has independent solutions $x_1 = e^{-t/2}\cos 3t$ and $x_1 = e^{-t/2}\sin 3t$

2.2.4 Applications

This section focuses on applications to damped spring-mass systems and RCL circuits. We work in the context of mechanical systems, but because of the mechanical-electrical analogy, all of the concepts apply to circuits as well.

There is suggestive terminology used in engineering and physics to describe the motion of a spring–mass system with damping, governed by the equation

$$mx'' + \gamma x' + kx = 0.$$

The characteristic equation is

$$m\lambda^2 + \gamma\lambda + k = 0,$$

with roots, or eigenvalues,

$$\lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}.$$

The first observation is that *the physical coefficients* m, γ, *and* k *are all nonnegative*. This restriction on the coefficients places restrictions on the eigenvalues and their signs.

(1) If $\gamma^2 - 4mk > 0$, the eigenvalues are real and distinct and they are *both negative*. The resulting solution decays to zero and the damped spring-mass system is called **overdamped**. In an overdamped system, the damping constant dominates the stiffness of the spring and the size of the mass.

(2) If $\gamma^2 = 4mk$ the eigenvalues are real and equal. They are negative and the solution decays to zero. In this special case the system is called **critically damped**. Damping balances mass and the stiffness of the spring.

(3) If $\gamma^2 - 4mk < 0$ the eigenvalues are complex. The eigenvalues have negative real part $-\gamma/2m$, and the solution is a **decaying oscillation**, or **underdamped**. The mass and the spring stiffness dominate damping. If $\gamma = 0$, there is no dissipation in the system and the eigenvalues are purely imaginary, giving an **oscillatory** solution. See Figure 2.3.

Here we often use the terminology "decaying oscillation" rather than "underdamped," and "damped" for both the overdamped and critically damped cases.

Example 2.11

(a) In the differential equation

$$2x'' + x' + 3x = 0$$

we have $\gamma^2 - 4mk = 1 - 4(2)(3) = -23 < 0$, so the solution is a *decaying oscillation*. The eigenvalues are complex, or

$$\lambda = -\frac{1}{4} \pm i\frac{\sqrt{23}}{4},$$

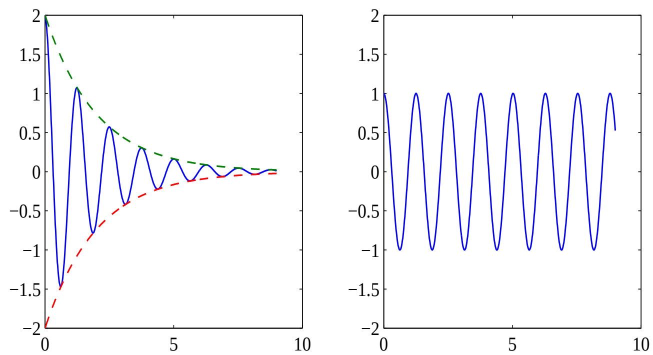

Figure 2.3 On the left, a decaying oscillation $x = 2e^{-t/2}\cos(5t)$; the exponential factor $2e^{-t/2}$ forms an envelop (dashed) of the oscillatory term. On the right, a pure oscillation $x = \cos(5t)$.

and the general solution is

$$x(t) = c_1 e^{-t/4} \cos \frac{\sqrt{23}}{4} t + c_2 e^{-t/4} \sin \frac{\sqrt{23}}{4} t.$$

(b) The differential equation

$$3x'' + 5x = 0$$

has no damping term ($\gamma = 0$) so the solution is purely *oscillatory*. The eigenvalues are

$$\lambda = \pm i\sqrt{\frac{5}{3}}$$

and the general solution is

$$x(t) = c_1 \cos \sqrt{\frac{5}{3}} t + c_1 \sin \sqrt{\frac{5}{3}} t. \qquad \square$$

Phase-Amplitude form

In the case of purely imaginary eigenvalues, we show from elementary trigonometry that the general solution

$$x(t) = c_1 \cos \beta t + c_2 \sin \beta t$$

can be written simply as

$$x(t) = A \cos(\beta t - \varphi),$$

where A is the **amplitude** and φ is the **phase**. This form is called the **phase–amplitude form** of the general solution. Written in this form, A and φ play the role of the two arbitrary constants, instead of c_1 and c_2. We now show that that all these constants are related by the formulas

$$A = \sqrt{c_1^2 + c_2^2}, \quad \varphi = \arctan \frac{c_2}{c_1}.$$

This is because the cosine of difference expands to

$$A\cos(\beta t - \varphi) = A\cos(\beta t)\cos\varphi + A\sin(\beta t)\sin\varphi.$$

Comparing this expression to $c_1 \cos\beta t + c_2 \sin\beta t$, gives

$$A\cos\varphi = c_1, \quad A\sin\varphi = c_2.$$

Squaring and adding this last set of equations determines A, and dividing the set of equations determines φ, both given as above. Note that the equation above for φ is valid in the first and fourth quadrants and occurs when $c_1 \geq 0$. If $c_1 < 0$, in the second and third quadrants, then we add π, that is $\varphi = \arctan c_2/c_1 + \pi$.

Sums of sines and cosines are difficult to plot without a calculator. But, we can write the phase-amplitude form as

$$x(t) = A\cos(\beta t - \varphi) = A\cos\left[\beta\left(t - \frac{\varphi}{\beta}\right)\right],$$

which easily plots as a shifted cosine function of frequency β and amplitude A. The amount of the shift, φ/β, is called the **phase shift**. The period of oscillation is $2\pi/\beta$ (in units of time). The frequency β is also called the **natural frequency** of the oscillator.

Example 2.12

For a spring-mass system with mass $m = 1$ kg and spring with stiffness $k = 5$ N/m, the initial position is 2 m and initial velocity is 1 m/s. The initial value problem is

$$x'' + 5x = 0, \quad x(0) = 2, \; x'(0) = 1.$$

The eigenvalues are $\lambda = \pm\sqrt{5}\, i$, so a fundamental set of solutions is $\cos\sqrt{5}t, \; \sin\sqrt{5}t$. The solution to the initial value problem is easily found to be

$$x(t) = 2\cos\sqrt{5}t + \frac{1}{\sqrt{5}}\sin\sqrt{5}t.$$

So, $c_1 = 2$ and $c_2 = 1/\sqrt{5}$. To determine the phase–amplitude form of the solution, we note

$$A = \sqrt{4 + \frac{1}{5}} = \sqrt{\frac{21}{5}}, \quad \varphi = \arctan\left(\frac{1}{2\sqrt{5}}\right) \approx 0.22,$$

because φ is in the first quadrant. Thus, the phase–amplitude form of the solution is

$$x(t) = \sqrt{\frac{21}{5}}\cos(\sqrt{5}t - 0.22) = \sqrt{\frac{21}{5}}\cos\left(\sqrt{5}(t - 0.098)\right).$$

The phase shift is $0.22/\sqrt{5} = 0.098$. The solution plots as the curve $\cos(\sqrt{5}t)$ shifted to the right by 0.098 and stretched vertically by the amplitude $\sqrt{21/5} = 2.049$. □

Example 2.13

The differential equation

$$x'' + 2x' + 5x = 0$$
$$x(0) = 1, \quad x'(0) = 3,$$

models a damped spring–mass system with $m = 1$, $\gamma = 2$, and $k = 5$. Initially it is displaced 1 unit and then given an initial velocity of 3 units. The characteristic equation is $\lambda^2 + 2\lambda + 5 = 0$. The quadratic formula gives complex roots $\lambda = -1 \pm 2i$. Therefore the general solution is

$$x(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t,$$

representing a decaying oscillation. We can use the phase–amplitude form to rewrite the oscillator part and obtain

$$x(t) = Ae^{-t}\cos(2t - \varphi).$$

The initial conditions are $x(0) = 1$, $x'(0) = 3$. We can use these conditions directly to determine either c_1 and c_2 in the first form of the solution, or A and φ in the phase–amplitude form. Going the former route, we apply the first condition to get

$$x(0) = c_1 = 1.$$

To apply the other initial condition we need the derivative. We get

$$x'(t) = -2c_1e^{-t}\sin(2t) - c_1e^{-t}\cos(2t) + 2c_2e^{-t}\cos(2t) - c_2e^{-t}\sin(2t).$$

Then

$$x'(0) = -c_1 + 2c_2 = 3.$$

Therefore $c_2 = 2$. The amplitude of the oscillatory part is

$$A = \sqrt{1^1 + 2^2} = \sqrt{5},$$

and the phase is

$$\varphi = \arctan 2 \approx 1.107 \text{ radians}.$$

In phase–amplitude form the solution is

$$x(t) = \sqrt{5}e^{-t}\cos(2t - 1.107).$$

The oscillatory part has natural frequency 2 and the period is π. The phase has the effect of translating the $\cos 2t$ term by $1.107/2 = 0.554$, which is the phase shift. □

EXERCISES

1. Find the solution to the initial value problem $x'' + x' + x = 0, \quad x(0) = x'(0) = 1$, and write it in phase-amplitude form.

2. A damped spring–mass system is modeled by the initial value problem

$$x'' + 0.125x' + x = 0, \quad x(0) = 2, \quad x'(0) = 0.$$

(a) Find the solution and sketch its graph over the time interval $0 \le t \le 50$.
(b) If the solution is written in the form $x(t) = Ae^{-t/16}\cos(\omega t - \varphi)$, find A, ω, and φ.

3. For which values of the parameters k (if any) will the solutions to $x'' + x' + kx = 0$ oscillate with no decay (i.e., be periodic)? Oscillate with decay? Decay without oscillations?

4. An RCL circuit has equation $LI'' + I' + I = 0$. Characterize the types of current responses that are possible, depending upon the value of the inductance L.

5. An oscillator with damping is governed by the equation $x'' + 3\delta x' + \kappa x = 0$, where δ and κ are positive parameters. Plot the set of points in the $\delta\kappa$ plane where the system is critically damped.

6. Describe the current response $I(t)$ of a LC circuit with $L = 5$ henrys, $C = 2$ farads, with $I(0) = 1$, $I'(0) = 1$. Express your answer in phase-amplitude form.

2.3 Nonhomogeneous Equations

In the last section we solved the **homogeneous equation**

$$ax'' + bx' + cx = 0. \tag{2.11}$$

Now we consider the **nonhomogeneous equation**

$$ax'' + bx' + cx = f(t), \tag{2.12}$$

where a prescribed term $f(t)$, called a **source term** or **forcing term**, is included on the right side. In mechanics $f(t)$ represents an applied, time-dependent force such as an imposed magnetic field driving the mass, and $x(t)$ represents the output, or displacement, resulting from that force.

Engineers and scientists regard equation (2.12) as an **input-output** system. In an RCL circuit the system is defined by parameters L, R, and C; to each applied voltage $E(t)$, or emf, which is the input, the response of the circuit is the solution $Q(t)$ of the nonhomogeneous equation

$$LQ'' + RQ' + \frac{1}{C}Q = E(t).$$

$Q(t)$ is the *input response*. Input-output systems are diagrammed as in Figure 2.4.

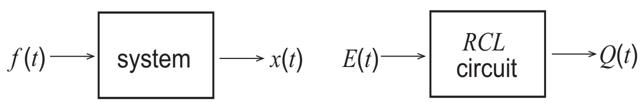

Figure 2.4 The differential equation (2.12) as an input-output system (left). An RCL circuit with an emf $E(t)$ as an input and the charge $Q(t)$ on the capacitor as the output (right).

There is a general structure theorem, analogous to the case with first-order linear equations, that specifies the *form* of the solution to the nonhomogeneous equation (2.12), and it gives a prescription on how to solve such equations.

Theorem 2.14

(**General Solution**) The general solution of the nonhomogeneous equation (2.12) is given by the sum of the general solution to the homogeneous equation (2.11) and any specific solution to the nonhomogeneous equation. That is, the general solution to (2.12) is

$$x(t) = c_1x_1(t) + c_2x_1(t) + x_p(t), \tag{2.13}$$

where $x_1(t)$ and $x_2(t)$ are two independent solutions to (2.11) and $x_p(t)$ is any solution whatsoever to (2.12). □

This result is very easy to show using Theorem 2.4. If $x(t)$ is any solution whatsoever of (2.12), and $x_p(t)$ is a particular solution, then it is easily checked that $x(t) - x_p(t)$ must satisfy the homogeneous equation (2.11). Therefore, by Theorem 2.4, $x(t) - x_p(t) = c_1x_1(t) + c_2x_1(t)$, which is (2.13). □

Notation. It is useful to introduce the notation $x_h(t)$ for general the solution to the homogeneous equation (2.11), that is,

$$x_h(t) = c_1x_1(t) + c_2x_1(t).$$

We call this the **homogeneous solution**; in some texts, it is called the complimentary solution. The specific solution $x_p(t)$ to the nonhomogeneous equation is called a **particular solution**. The general solution to (2.12) can therefore be written

$$x(t) = x_h(t) + x_p(t),$$

which is the sum of the homogeneous solution and the particular solution.

2.3.1 Undetermined Coefficients

We already know how to find the solution to the homogeneous equation (2.11), so we need techniques to find a specific solution $x_p(t)$ to the nonhomogeneous equation (2.12):

$$ax'' + bx' + cx = f(t).$$

One method that works for many equations is simply to make an assumption, or judicious *guess*, based on the form of the source term $f(t)$. Officially, this method is called the method of **undetermined coefficients** because we eventually have to find numerical coefficients in our choice for $x_p(t)$ when we substitute it into the differential equation. The method works most of the time, because all the terms on the left side of (2.12) must eventually add up to give $f(t)$. Therefore the particular solution must have roughly the same form as $f(t)$. The method is successful for forcing terms that occur in many mechanical and circuit problems, such as *constants*, *exponential functions*, *sines* and *cosines*, *polynomials*, and *sums* and *products* of these common functions.

Table 2.2 indicates the trial form of the particular solution $x_p(t)$ depending on the form of the source term in the differential equation.

There is only one sticky issue. If a trial guess $x_p(t)$ of the form of a particular solution coincides with one of the fundamental solutions $x_1(t)$, $x_2(t)$ of

Form of source function $f(t)$	Trial form of particular solution $x_p(t)$
α	A
$\alpha e^{\beta t}$	$Ae^{\beta t}$
Polynomial of degree n	$A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0$
$\alpha \sin \omega t$; $\alpha \cos \omega t$	$A \sin \omega t + B \cos \omega t$
$\alpha e^{rt} \sin \omega t$; $\alpha e^{rt} \cos \omega t$	$e^{rt}(A \sin \omega t + B \cos \omega t)$

Table 2.2 A table showing the form $x_p(t)$ of an initial trial particular solution to the nonhomogeneous equation $ax'' + bx' + cx = f(t)$ based on the form of the source $f(t)$. The Greek letters in the left column denote known constants in the source term $f(t)$, while the uppercase Latin letters in the right column denote coefficients to be determined in a trial form.

the homogeneous equation, then that particular solution will itself satisfy the homogeneous equation and therefore cannot equal $f(t)$. Then we must modify the trial guess. All that is required is to multiply our trial guess by a positive power of t. Stated formally, we have:

Remark 2.15

Caveat. *If a term in the initial trial guess for a particular solution x_p duplicates one of the fundamental solutions for the homogeneous equation, then modify the guess by multiplying by the smallest power of t that eliminates the duplication.*

Several examples illustrate these ideas. As you will soon learn, the analytic calculations to determine the unknown constants can often be tedious. But the technique is useful to mathematically understand some essential physical concepts, such as the origin of resonance. Certainly, solutions to complicated problems can be done using one of the many available software packages.

Example 2.16

In this first example we ignore the differential equation itself and give some simple examples of determining an initial, trial particular solution, based only on the form of the forcing function $f(t)$.

(a) If $f(t) = 6$, take $x_p(t) = A$.

(b) If $f(t) = 3t^2 - 3$, take $x_p(t) = At^2 + Bt + C$.

(c) If $f(t) = 12 \sin 3t$, take $x_p(t) = A \cos 3t + B \sin 3t$.

(d) If $f(t) = -3e^{4t}$, take $x_p(t) = Ae^{4t}$.

(e) If $f(t) = 2te^{-3t}$, take $x_p(t) = (At + B)e^{-3t}$.

(f) If $f(t) = e^{-3t} \sin 2t$, take $x_p(t) = e^{-3t}(A \cos 2t + B \sin 2t)$.

(g) If $f(t) = te^{-3t} \sin 2t$, take $x_p(t) = (At + B)e^{-3t}(C \cos 2t + D \sin 2t)$. □

Example 2.17

Find the general solution to the following equation with a source term $f(t) = 4e^{-5t}$:

$$x'' + 3x = 4e^{-5t}. \tag{2.14}$$

First we find the *homogeneous solution* $x_h(t)$. The eigenvalues are $\lambda = \pm\sqrt{3}\,i$, purely imaginary, and the general solution to the homogeneous equation is

$$x_h(t) = c_1 \cos \sqrt{3}\,t + c_2 \sin \sqrt{3}\,t.$$

Next we find the *particular solution* $x_p(t)$. The two terms on the left side of the (2.14) must add to give $f(t)$, so it is plausible to assume a trial particular solution $x_p(t) = Ae^{-5t}$, where A is to be determined. (See the second entry of Table 2.2.) Moreover, this form of x_p does not coincide with either of the two fundamental solutions, $\cos \sqrt{3}\,t,\ \sin \sqrt{3}\,t$. So we substitute $x_p(t)$ into the differential equation to get $25Ae^{-5t} + 3Ae^{-5t} = 4e^{-5t}$, or $A = 1/7$. A particular solution is therefore

$$x_p(t) = \frac{1}{7}e^{-5t}.$$

Therefore, the general solution to the nonhomogeneous equation (2.14) is $x(t) = x_h(t) + x_p(t)$. By the way, if there are initial conditions, apply them to general solution at the end of the calculation and not to only x_h. □

Example 2.18

Solve the differential equation

$$x'' - x' + 7x = 5t - 3.$$

The homogeneous solution is

$$x_h(t) = c_1 e^{-t/2} \cos(\sqrt{27/2}\,t) + c_2 e^{-t/2} \sin(\sqrt{27/2}\,t).$$

The right side, $f(t) = 5t - 3$, is a polynomial of degree 1 so we try $x_p = At + B$, a polynomial of degree 1. See the third entry in Table 2.2. Substituting, we get $-A + 7(At + B) = 5t - 3$. Equating like terms gives $-A + 7B = -3$ and $7A = 5$. Therefore, $A = 5/7$ and $B = -16/49$, and a particular solution to the given equation is

$$x_p(t) = \frac{5}{7} - \frac{16}{49}t.$$

The general solution is $x(t) = x_h(t) + x_p(t)$. □

Example 2.19

Solve the differential equation

$$x'' - x = 5e^{-t}.$$

The characteristic polynomial for the homogeneous equation is $\lambda^2 - 1 = 0$, having eigenvalues $\lambda = -1, +1$. Therefore a fundamental solution set is $e^{-t},\ e^t$, and the homogeneous solution is

$$x_h(t) = c_1 e^{-t} + c_2 e^t.$$

To find a particular solution, take a trial solution $x_p(t) = Ae^{-t}$. But this duplicates one of the fundamental solutions. If we substitute Ae^{-t} into the equation we get zero, not $5e^{-t}$! So we must modify our trial function by multiplying by t, giving $x_p(t) = Ate^{-t}$. The duplication is eliminated because this is not a solution to the homogeneous equation. So we substitute in the equation to obtain

$$[Ate^{-t} - 2Ae^{-t}] - Ate^{-t} = 5e^{-t},$$

or $-2Ae^{-t} = 5e^{-t}$. Thus $A = -5/2$ and the particular solution is $x_p(t) = -\frac{5}{2}te^{-t}$. The general solution is

$$x(t) = c_1 e^{-t} + c_2 e^t - \frac{5}{2} te^{-t}. \quad \square$$

First-order equations. The method of undetermined coefficients works equally well for first-order equations of the form $x' + px = f(t)$, where p is constant. The homogeneous equation has characteristic polynomial $\lambda + p = 0$, giving the eigenvalue $\lambda = -p$. Its solution is therefore $x_h(t) = ce^{-pt}$. Provided $f(t)$ has the form of the entries in Table 2.2, a particular solution $x_p(t)$ is found exactly as for second-order equations: make a trial guess and substitute into the equation to determine the coefficients in the guess. Modify the trial if it duplicates the homogeneous solution e^{-pt}. The general solution to the nonhomogeneous equation is then $x(t) = ce^{-pt} + x_p(t)$.

Example 2.20

Consider the equation

$$x' - 3x = t^2.$$

The homogeneous solution is $x_h = ce^{3t}$. The forcing term is a quadratic, so a trial guess for a particular solution is a general quadratic,

$$x_p = At^2 + Bt + C.$$

No term duplicates the homogeneous solution. Substituting this into the equation and equating coefficients of like terms gives three algebraic equations for A, B, and C, namely

$$-3A = 1, \quad 2A - 3B = 0, \quad B - 3C = 0.$$

Thus $A = -\frac{1}{3}$, and $B = -\frac{2}{9}$, and $C = -\frac{2}{27}$. Consequently, the general solution is

$$x(t) = ce^{3t} - \frac{1}{3}t^2 - \frac{2}{9}t - \frac{2}{27}.$$

This calculation was simple. Using integrating factors would require a more complicated integration by parts to obtain the solution. □

Example 2.21

Find the solution to the initial value problem:

$$x'' + 2x = \sin 3t, \quad x(0) = 1, \quad x'(0) = -1.$$

This is a spring-mass oscillator equation with a periodic forcing function having frequency 3 and period $2\pi/3$. The homogeneous solution, or free response, is

$$x_h(t) = c_1 \cos \sqrt{2}\, t + c_2 \sin \sqrt{2}\, t.$$

It oscillates with natural frequency $\beta = \sqrt{2}$. To find a particular solution we try (Table 2.2) $x_p = A \sin 3t + B \cos 3t$. Neither of these terms coincide with those in the fundamental solution set. Therefore, upon substituting,

$$-9A \sin 3t - 9B \cos 3t + 2A \sin 3t + 2B \cos 3t = \sin 3t.$$

Equating like terms gives $-9A + 2A = 1$ and $B = 0$ (there are no cosine terms on the right side). Hence $A = -1/7$ and a particular solution is

$$x_p = -\frac{1}{7} \sin 3t.$$

The general solution is therefore

$$x(t) = c_1 \cos \sqrt{2}\, t + c_2 \sin \sqrt{2}\, t - \frac{1}{7} \sin 3t.$$

Now we apply the initial conditions. Clearly, $x(0) = c_1 = 1$. Then,

$$x'(t) = -\sqrt{2} \sin \sqrt{2}\, t + c_2 \sqrt{2} \cos \sqrt{2}\, t - \frac{3}{7} \cos 3t,$$

giving $x'(0) = c_2\sqrt{2} - \frac{3}{7} = -1$, or $c_2 = -4/7\sqrt{2}$. The unique solution to the IVP is

$$x(t) = \cos \sqrt{2}\, t - \frac{4}{7\sqrt{2}} \sin \sqrt{2}\, t - \frac{1}{7} \sin 3t.$$

This solution is a superposition of an oscillation of frequency $\sqrt{2}$ in the homogeneous solution and an oscillation of frequency 3 in the particular solution. A plot showing the interaction is shown in Figure 2.5. This models, for example, an LC circuit with natural frequency $\sqrt{2}$ with an emf driving the circuit with frequency 3. □

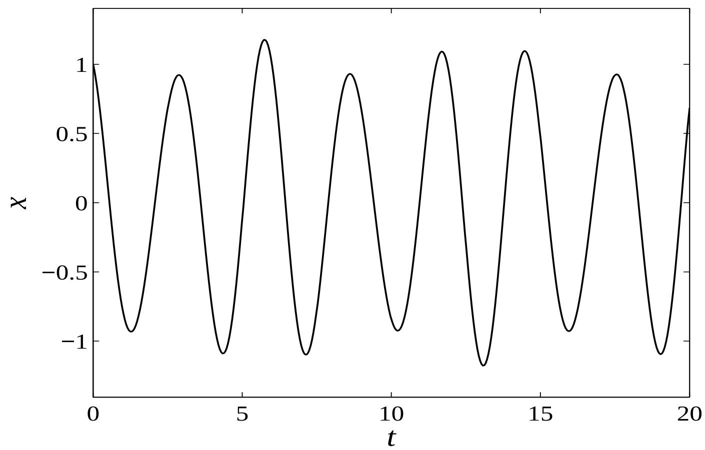

Figure 2.5 Plot of $x(t) = \cos\sqrt{2}\,t - \frac{4}{7\sqrt{2}}\sin\sqrt{2}\,t - \frac{1}{7}\sin 3t$.

Remark 2.22

(**Superposition**) If the source term in a differential equation is a sum $f_1(t) + f_2(t)$ of two sources, then we can break up the problem into two separate equations if needed, that is,

$$ax'' + bx' + cx = f_1(t) \quad \text{and} \quad ax'' + bx' + cx = f_2(t) \tag{2.15}$$

and find a particular solution for each. Then a particular solution of the equation

$$ax'' + bx' + cx = f_1(t) + f_2(t)$$

is the sum of the particular solutions to (2.15). □

Example 2.23

Find the general solution to the nonhomogeneous equation

$$x'' + 3x' + 3x = 6e^{-2t} + 4.$$

The homogeneous equation has characteristic polynomial $\lambda^2 + 3\lambda + 3 = 0$, which has roots $\lambda = -3/2 \pm i\sqrt{3}/2$. Thus the solution to the homogeneous equation is

$$x_h(t) = c_1 e^{-3t/2} \cos \frac{\sqrt{3}}{2} t + c_2 e^{-3t/2} \sin \frac{\sqrt{3}}{2} t.$$

Next we find a particular solution to the nonhomogeneous equation. Take $f_1(t) = 6e^{-2t}$, and try $x_{p1} = Ae^{-2t}$. Substituting this trial function into the nonhomogeneous equation gives, after canceling e^{-2t}, the equation $4A - 6A + 3A = 6$. Thus $A = 1$ and a particular solution to the nonhomogeneous equation is $x_{p1} = e^{-2t}$. Next take $f_2(t) = 4$. A particular solution is quickly a constant, or $x_{p2} = 4/3$. The general solution to the original nonhomogeneous equation is

$$x(t) = c_1 e^{-3t/2} \cos \frac{\sqrt{3}}{2} t + c_2 e^{-3t/2} \sin \frac{\sqrt{3}}{2} t + e^{-2t} + \frac{4}{3}.$$

Note that $\lim_{t \to \infty} x(t) = \frac{4}{3}$. Eventually the transients of the system decay away and it settles into a constant steady state. □

Example 2.24

(**RCL circuit**) Consider an RCL circuit with $R = 2$, $L = C = 1$, with the current being driven by an electromotive force of $2 \sin 3t$. The circuit equation for the charge $Q(t)$ across the capacitor is

$$Q'' + 2Q' + Q = 2 \sin 3t.$$

For initial data take

$$Q(0) = 4, \quad Q'(0) = 0.$$

The homogeneous equation has characteristic equation $\lambda^2 + 2\lambda + 1 = 0$ with a double root $\lambda = -1, -1$. The homogeneous solution is therefore

$$Q_h = e^{-t}(c_1 + c_2 t).$$

Regardless of the values of the constants, this decays away as time t increases; this part of the solution is called the **transient response** of the circuit. To find a particular solution we use undetermined coefficients and assume it has the form

$$Q_p = A \sin 3t + B \cos 3t.$$

Substituting into the nonhomogeneous equation gives a pair of linear equations for A and B,

$$-4A - 3B = 1, \quad 7A - 9B = 0.$$

We find $A = -0.158$ and $B = -0.123$. Therefore the general solution is

$$Q(t) = e^{-t}(c_1 + c_2 t) - 0.158 \sin 3t - 0.123 \cos 3t.$$

Now apply the initial conditions. Easily, $Q(0) = 4$ implies $c_1 = 4.123$. Next we find $Q'(t)$ so that we can apply the condition $Q'(0) = 0$. Leaving this as an exercise, we find $c_2 = 4.597$. Therefore, the voltage on the capacitor is

$$Q(t) = e^{-t}(4.123 + 4.597t) - 0.158 \sin 3t - 0.123 \cos 3t.$$

As we observed, the first term, or transient, decays as time increases. Therefore we are left with only the oscillatory particular solution $-0.158 \sin 3t - 0.123 \cos 3t$, which takes over in time. This is called the **steady-state response** of the circuit (Figure 2.6). □

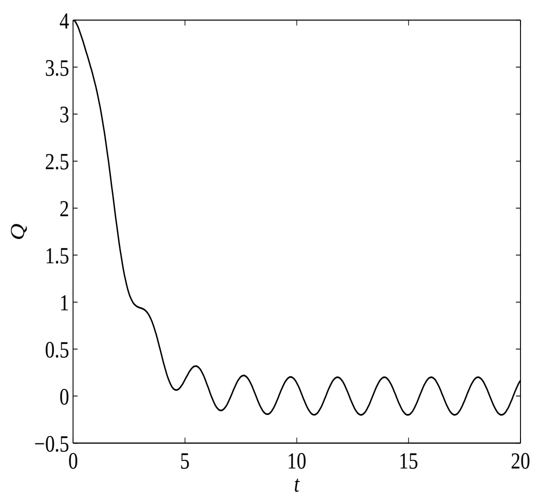

Figure 2.6 A plot of the voltage $Q(t)$ in Example 2.24. Initially there is a transient caused by the initial conditions. In time it decays away and is replaced by a steady-state response, an oscillation, that is caused by the forcing term.

Example 2.25

Consider an LC circuit with a periodic emf $\sin 3t$ of frequency 3:

$$Q'' + 9Q = \sin 3t,$$

where Q is the charge on the capacitor. The rules dictate the trial particular solution $Q_p = A\sin 3t + B\cos 3t$. Substituting into the differential equation yields

$$-9A\sin 3t - 9B\cos 3t + 9A\sin 3t + 9B\cos 3t = \sin 3t.$$

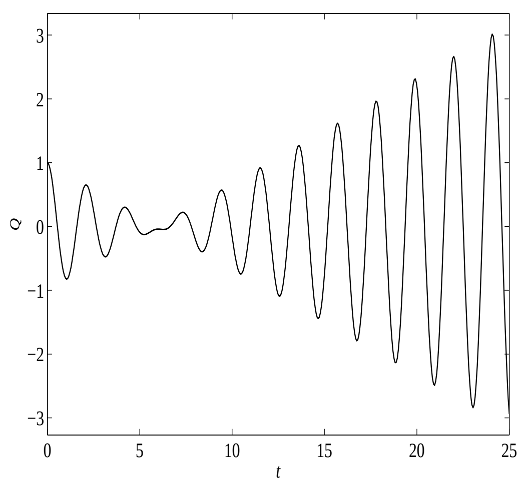

Figure 2.7 Solution $Q(t) = \cos 3t + \frac{1}{18}\sin 3t - \frac{1}{6}t\cos 3t$. The increasing amplitude of the oscillations is caused by driving the system at the same frequency as its natural frequency. This phenomenon is called resonance.

But the terms on the left cancel completely and we get $0 = \sin 3t$, an absurdity. The method failed! To review, this is because the homogeneous equation $Q'' + 9Q = 0$ has eigenvalues $\lambda = \pm 3i$, which lead to independent solutions $Q_1 = \sin 3t$ and $Q_2 = \cos 3t$. Each of these has natural frequency equal to 3. The forcing term $f(t) = \sin 3t$, which also has frequency 3, is not independent; it duplicates one of them and the method fails. The fact that we get 0 when we substitute our trial function into the equation is no surprise because it is a solution to the homogeneous equation. To remedy this problem we use the rule and modify the trial guess by multiplying by t. So we look for a particular solution of the form

$$Q_p = t(A\sin 3t + B\cos 3t).$$

Calculating the second derivative Q_p'' and substituting, along with Q_p, into the original equation leads to (show this!)

$$6A\cos 3t - 6B\sin 3t = \sin 3t.$$

Hence $A = 0$ and $B = -1/6$. So a particular solution is

$$Q_p = -\frac{1}{6}t\cos 3t,$$

and the general solution of the original nonhomogeneous equation is the homogeneous solution plus the particular solution, or

$$Q(t) = c_1\cos 3t + c_2\sin 3t - \frac{1}{6}t\cos 3t.$$

Next we apply the initial conditions to fix the arbitrary constants. If $Q(0) = 1$ and $Q'(0) = 0$, then it is easy to show that $c_1 = 1$ and $c_2 = 1/18$. So, the solution to the initial value problem is

$$Q(t) = \cos 3t + \frac{1}{18}\sin 3t - \frac{1}{6}t\cos 3t.$$

This solution is plotted in Figure 2.7. Observe that the solution to the homogeneous equation is oscillatory and remains bounded; the particular solution, however, oscillates with increasing amplitude because of the time factor t multiplying that term. This phenomenon is called **resonance**. It occurs because the forcing function $\sin 3t$ has frequency 3, which is the same as the natural frequency of the unforced system. This phenomenon is discussed in more detail in the next section. □

EXERCISES

1. Each of the following functions represents a source term $f(t)$ in a nonhomogeneous equation. State the *form* of an initial trial guess for a particular solution $x_p(t)$.

a) $3t^3 - 1$.

b) $3\cos t - 2\sin t$.

c) 12.

d) t^2e^{3t}.

e) $5\sin 7t$.

f) $e^{2t}\cos t + t^2$.

g) $te^{-t}\sin \pi t$.

h) $(t+2)\sin \pi t$.

i) $4t + 5e^{-t}$.

j) $5\sin(2t) + te^t$.

k) $t^3 + 1 - 4t\cos t$.

l) $-6 + 2e^{2t}\sin t$.

2. Find the general solution of the following nonhomogeneous equations:

   a) $x'' + 7x = te^{3t}$.
   b) $x'' - x' = 6 + e^{2t}$.
   c) $x' + x = t^2$.
   d) $x'' - 3x' - 4x = 2t^2$.
   e) $x'' + x = 9e^{-t}$.
   f) $x' + x = 4e^{-t}$.
   g) $x'' - 4x = \cos 2t$.
   h) $x'' + x' + 2x = t \sin 2t$.

3. Solve the initial value problem $x'' - bx' + x = \sin t, \quad x(0) = 0, \quad x'(0) = 0$, where b is a constant with $b < 1$.
4. Solve the initial value problem $x'' - 3x' - 40x = 2e^{-t},\ x(0) = 0,\ x'(0) = 1$.
5. Find the solution of $x'' - 2x' = 4$, $x(0) = 1$, $x'(0) = 0$.
6. An undamped spring–mass system is driven by an external force $\cos \sqrt{2}\, t$. The mass is $m = 1$ and the spring constant is $k = 2$. Initially, $x(0) = 0$ and $x'(0) = 1$. Find the general solution and plot it for $0 \le t \le 30$.
7. Find the particular solution to the equation $I'' + I' + 2I = \sin^2 t$? Hint: Use a trigonometry formula to rewrite the right side.
8. Solve $y'' + 2y' = e^{-t},\ \ y(0) = 1,\ y'(0) = 2$.
9. A mass of 5 grams is attached to a spring with stiffness $k = 2$. The system is driven by an external force of 10 dynes. Initially the mass is displaced 15 cm and given a velocity of 4 cm/sec. Find and plot the displacement of the mass for all times $t \ge 0$.
10. An LC circuit contains a 10^{-2} farad capacitor in series with an aging battery of $5e^{-2t}$ volts and an inductor of 0.4 henrys. At $t = 0$ both $Q = 0$ and $I = 0$. Find the charge $Q(t)$ on the capacitor and describe the response of the circuit in terms of transients and steady states.
11. An RCL circuit contains a battery generating 110 volts. The resistance is 16 ohms, the inductance is 2 henrys, and the capacitance is 0.02 farads. If $Q(0) = 5$ and $I(0) = 0$, find the charge $Q(t)$ response of the circuit. Identify the transient solution and the steady-state response.
12. An RCL circuit contains a aging battery generating $10e^{-t/100}$ volts. The resistance is 100 ohms, the inductance is 2 henrys, and the capacitance is 0.001 farads. If $Q(0) = Q'(0) = 0$, find the charge $Q(t)$ on the capacitor for $t > 0$ and sketch its graph. When does the maximum charge occur?

2.3.2 Resonance

In this section we expand on the concept of **resonance**. See Example 2.25. The phenomenon of *resonance* is a key characteristic of vibrating systems. It occurs when the *frequency of a forcing term has the same frequency as the natural oscillations* in the system; it gives rise to large amplitude oscillations. Resonance can occur in circuits when a generator drives the system at its natural frequency, or it can occur in mechanical systems and structures where an external periodic force is applied at the same frequency as the system would naturally oscillate. The results could be disastrous, such as a blown circuit or a damaged building. Similarly, a bridge can have a natural frequency of oscillation and the wind can provide the forcing function. (See on-line videos of the Tacoma Narrows Bridge disaster.)

Example 2.26

Consider an LC circuit driven by a sinusoidal voltage source of frequency ω. If $L = 1$ and $\omega_0^2 = 1/C$, the governing equation for the charge on the capacitor has the form

$$Q'' + \omega_0^2 Q = \sin \omega t. \tag{2.16}$$

Assume that $\omega_0 \neq \omega$ and take initial conditions

$$Q(0) = 0, \quad Q'(0) = 1.$$

The homogeneous equation has general solution

$$Q_h = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t,$$

which represents oscillations of *natural frequency* ω_0. A particular solution has the form $Q_p = A \sin \omega t$. Substituting into the equation gives $A = 1/(\omega_0^2 - \omega^2)$. Therefore the general solution of (2.16) is

$$Q(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{1}{\omega_0^2 - \omega^2} \sin \omega t. \tag{2.17}$$

At $t = 0$ we have $Q = 0$ and so $c_1 = 0$. Also $Q'(0) = 1$ gives

$$c_2 = -\frac{\omega + \omega_0(\omega_0^2 - \omega^2)}{\omega_0^2 - \beta^2}.$$

Therefore the response of the circuit is

$$Q(t) = -\frac{\omega + \omega_0(\omega_0^2 - \omega^2)}{\omega_0^2 - \omega^2} \sin \omega_0 t + \frac{1}{\omega_0^2 - \omega^2} \sin \omega t. \tag{2.18}$$

This solution shows that the charge response is a sum of two oscillations of different frequencies. If the forcing frequency ω is close to the natural frequency ω_0, then the amplitude is bounded, but it is obviously large because of the factor $\omega_0^2 - \omega^2$ occurring in the denominator. Thus the system has large oscillations when ω is close to ω_0. □

Example 2.27

In the previous example, what happens if $\omega = \omega_0$? Then the general solution in (2.17) is invalid because of division by zero; thus we have to re-solve the problem. The circuit equation is

$$Q'' + \omega_0^2 Q = \sin \omega_0 t, \tag{2.19}$$

where the circuit is forced at the *same frequency* as its natural frequency. The homogeneous solution is the same as before, but the particular solution now has the form

$$Q_p(t) = t(A \sin \omega_0 t + B \cos \omega_0 t),$$

with a factor of t multiplying the terms. Therefore the general solution of (2.19) has the form

$$Q(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + t(A \sin \omega_0 t + B \cos \omega_0 t).$$

Without actually determining the constants, we can infer the nature of the response. Because of the t factor in the particular solution, the amplitude of the oscillatory response $Q(t)$ grows in time. This is the phenomenon of *pure resonance*, or mathematical resonance. It occurs when the frequency of the external force is the same as the natural frequency of the system. □

The previous example is an ideal case and physically unreasonable. All circuits have resistance, or dissipation, even though it may be small. We ask what happens if we include a small damping term in the circuit and still force it at its natural frequency.

Example 2.28

Consider

$$Q'' + 2\sigma Q' + 2Q = \sin \sqrt{2}\, t,$$

where 2σ is a small. The homogeneous equation $Q'' + 2\sigma Q' + 2Q = 0$ has solution

$$Q_h(t) = e^{-\sigma t} \left(c_1 \cos \sqrt{2 - \sigma^2}\, t + c_2 \sin \sqrt{2 - \sigma^2}\, t \right).$$

Now the particular solution has the form

$$Q_p(t) = A\cos\sqrt{2}\,t + B\sin\sqrt{2}\,t,$$

where A and B are constants. The solution, or response, of the circuit is

$$Q(t) = e^{-\sigma t}\left(c_1\cos\sqrt{2-\sigma^2}\,t + c_2\sin\sqrt{2-\sigma^2}\,t\right) + A\cos\sqrt{2}\,t + B\sin\sqrt{2}\,t.$$

The transient is a decaying oscillation of frequency $\sqrt{2-\sigma^2}$, and the steady-state response is periodic of frequency $\sqrt{2}$. The solution will remain bounded, but its amplitude will be large if σ is very small. See Figure 2.8. □

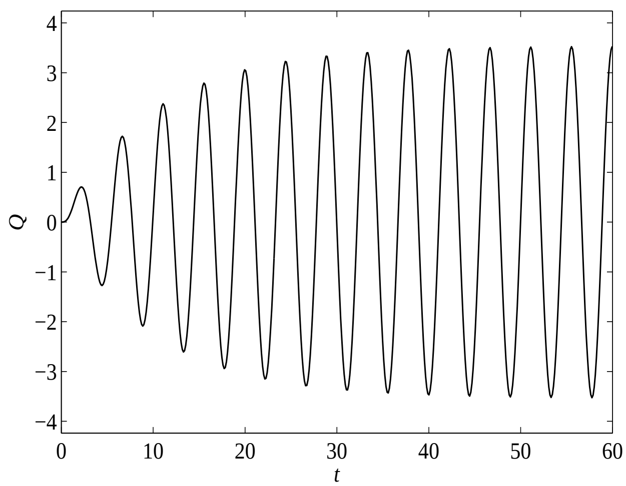

Figure 2.8 Plot of the solution of $Q'' + 0.2Q' + 2Q = \sin(\sqrt{2}\,t)$ with zero initial conditions. The system is driven at a frequency equal to the natural frequency, and there is small damping.

EXERCISES

1. Plot the solution (2.18) for several different values of β and ω. Include values where these two frequencies are close.
2. Find the solution in Example 2.27 if the initial conditions are $Q(0) = Q'(0) = 0$.
3. Find the *form* of the general solution of the equation $I'' + 16I = \cos 4t$.

4. Consider a general LC circuit with input voltage $V_0 \sin \beta t$. If β and the capacitance C are known, what value of the inductance L causes pure resonance?

5. An undamped spring–mass system with $m = 4$ and and stiffness k is forced by a sinusoidal function $412 \sin 5t$. What value of k causes pure resonance?

6. Consider a spring–mass system with small damping and driven by a cosine force:
$$x'' + 0.01x' + 4x = \cos 2t, \quad x(0) = 0,\ x'(0) = 0.$$
Find the solution and plot the result.

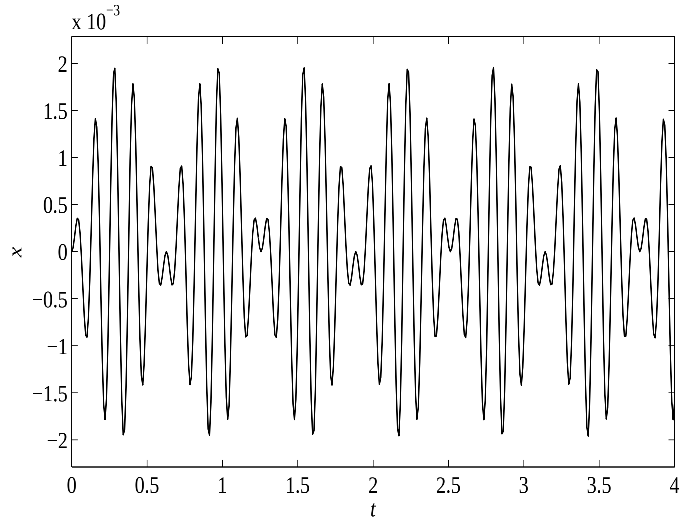

Figure 2.9 Plot of the solution $x(t) = 10^{-3}(\cos(45t) - \cos(55t))$ to Exercise 7(c) showing the phenomenon of beats.

7. Consider the equation
$$x'' + \omega^2 x = \cos \beta t.$$

   a) Find the solution when the initial conditions are $x(0) = x'(0) = 0$ when $\omega \neq \beta$.

   b) Use the trigonometric identity $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$ to write the solution as a product of sines.

   c) Take $\omega = 55$ and $\beta = 45$ and plot the solution from part (b) on the time interval $[0, 4]$. The solution can be interpreted as a high frequency response contained in a low frequency amplitude envelope. We say the high frequency is *modulated* by the low frequency. This is the phenomenon of **beats**. What is the high frequency and low frequency modulation? A plot of the solution is shown in Figure 2.9.

2.4 Equations with Variable Coefficients

Second-order linear equations with given *variable coefficients* $p(t)$ and $q(t)$ have the form

$$x'' + p(t)x' + q(t)x = f(t). \tag{2.20}$$

Except for a few cases, these equations cannot be solved in analytic form using familiar functions. Even the simplest equation, **Airy's equation**,

$$x'' - tx = 0,$$

requires a new class of transcendental functions, called *Airy functions*, to characterize the solutions. Nonetheless, there is a well developed theory for variable coefficient equations, and we list some of the main results.

We assume that the coefficients $p(t)$ and $q(t)$, as well as the forcing term $f(t)$, are continuous functions on the interval I of interest. The following properties are the same ones shared by second-order, constant coefficient equations studied in Section 2.2.

1. (**Existence-Uniqueness**) If I is an open interval and t_0 belongs to I, then the initial value problem

$$x'' + p(t)x' + q(t)x = f(t), \tag{2.21}$$

$$x(t_0) = a, \quad x'(t_0) = b, \tag{2.22}$$

has a unique solution on I.

2. (**The Homogeneous Equation**) If x_1 and x_2 are independent solutions of the associated homogeneous equation

$$x'' + p(t)x' + q(t)x = 0 \tag{2.23}$$

on an interval I, then

$$x(t) = c_1x_1(t) + c_2x_2(t)$$

for any constants c_1 and c_2 is a solution of (2.23) on the interval I and is called the **general solution** to (2.23). The general solution contains all solutions of the homogeneous equation for various choices of the constants.

3. (**Nonhomogeneous Equation**) All solutions to the nonhomogeneous equation (2.21) can be represented as the sum of the general solution to the homogeneous equation (2.23) and any particular solution $x_p(t)$ to the nonhomogeneous equation (2.21). In symbols,

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + x_p(t),$$

which is called the **general solution** to (2.21).

One difficulty, of course, is to find two independent solutions $x_1(t)$ and $x_2(t)$ to the homogeneous equation. This task is not easily accomplished for equations with variable coefficients. The method of writing down the characteristic polynomial, as we did for constant coefficient equations, *does not work*. A method that applies to many of these types of problems is called the *power series* method. Basically, the idea is to assume the solution has the form of a infinite power series, which we substitute it into the equation and determine relations for the coefficients. This topic is not covered in this text, and we refer to the references for a complete discussion. For example, see Brannan & Boyce (2011).

2.4.1 Cauchy–Euler Equations

One equation that can be solved analytically has the form

$$x'' + \frac{b}{t}x' + \frac{c}{t^2}x = 0,$$

or

$$t^2 x'' + btx' + cx = 0, \tag{2.24}$$

which is called a **Cauchy–Euler equation**. In each term the exponent on t coincides with the order of the derivative. Observe that we must avoid $t = 0$ in our interval of solution, because $p(t) = b/t$ and $q(t) = c/t^2$ are not continuous at $t = 0$. We try to find a solution of the form of a power function $x(t) = t^m$. Think about why this might work—each term in (2.24) will then have the same power of t, namely, t^m. Substituting gives the the **indicial equation**

$$m(m-1) + bm + c = 0,$$

which is a quadratic equation for for the exponent m.

There are three cases.

(1) If there are **two distinct real roots** m_1 and m_2, then we obtain two independent solutions t^{m_1} and t^{m_1}. Therefore the general solution is

$$x(t) = c_1 t^{m_1} + c_2 t^{m_2}.$$

(2) If there are **two equal roots** $m_1 = m_2 = m$, then t^m and $t^m \ln t$ are two independent solutions and the general solution is

$$x(t) = c_1 t^m + c_2 t^m \ln t.$$

(3) If the two **roots are complex conjugates**, $m = \alpha \pm i\beta$, we note, using the properties of logarithms, exponentials, and Euler's formula, that a complex solution is

$$t^m = t^{\alpha+i\beta} = t^\alpha t^{i\beta} = t^\alpha e^{\ln t^{i\beta}} = t^\alpha e^{i\beta \ln t} = t^\alpha[\cos(\beta \ln t) + i \sin(\beta \ln t)].$$

The real and imaginary parts of this complex function are therefore real solutions (proved earlier), so the general solution in the complex case is

$$x(t) = c_1 t^\alpha \cos(\beta \ln t) + c_2 t^\alpha \sin(\beta \ln t).$$

Figure 2.10 shows a graph of the function $\sin(5 \ln t)$, which is a type appearing in the solution in the complex case (3). Note that this function oscillates less and less as t gets large because $\ln t$ grows very slowly. As t nears zero it oscillates infinitely many times. Because of the scale, these rapid oscillations are not apparent on the plot. Initial conditions are never given at $t = 0$.

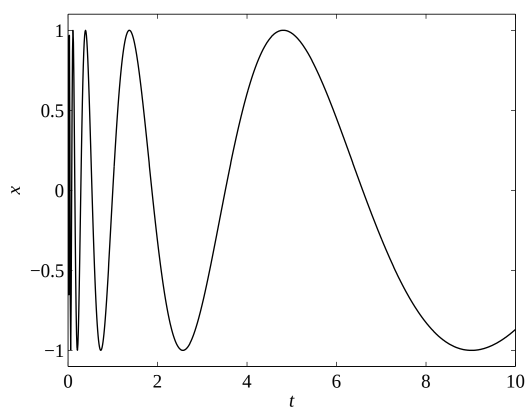

Figure 2.10 Plot of $x = \sin(5 \ln t)$, which oscillates infinitely many times near the origin.

Example 2.29

Consider the equation

$$t^2x'' + tx' + 9x = 0.$$

The indicial equation is $m(m-1)+m+9=0$, which has roots $m = \pm 3i$. The general solution is therefore

$$x(t) = c_1 \cos(3 \ln t) + c_2 \sin(3 \ln t). \qquad \square$$

Example 2.30

Consider the equation

$$x'' = \frac{2}{t}x'.$$

We can write this in Cauchy–Euler form as

$$t^2x'' - 2tx' = 0,$$

which has indicial equation $m(m-1) - 2m = 0$. The roots are $m = 0$ and $m = 3$. Therefore the general solution is

$$x(t) = c_1 + c_2t^3. \qquad \square$$

Example 2.31

Solve the initial value problem

$$t^2x'' + 3tx' + x = 0, \quad u(1) = 0, \quad x'(1) = 2.$$

The indicial equation is $m(m-1) + 3m + 1 = 0$, and it has a double root $m = -1$; so the general solution is

$$x(t) = \frac{c_1}{t} + \frac{c_2}{t} \ln t.$$

Now, $x(1) = c_1 = 0$ and so $x(t) = c_2\ (t \ln t)$. Taking the derivative, $x'(t) = c_2/(t^2(1 - \ln t))$. Then $x'(1) = c_2 = 2$. Hence, the solution to the initial value problem is

$$x(t) = \frac{2}{t} \ln t. \qquad \square$$

A. Cauchy (1789–1857) and L. Euler (1707–1783) were renown mathematicians who left an indelible mark on the history of mathematics and science. Their names are encountered often in advanced courses in mathematics, science, and engineering.

EXERCISES

1. Solve the following problems as indicated:

   a) $x'' = -\frac{1}{t^2}x$.

   b) $x'' = \frac{4}{t^2}x$.

   c) $t^2x'' + 3tx' = +x = 0$.

   d) $tx'' + 4x' + \frac{2}{t}x = 0$.

   e) $t^2x'' - 7tx' + 16x = 0$.

   f) $t^2x'' + 3tx' - 8x = 0$, $x(1) = 0$, $x'(1) = 2$.

   g) $t^2x'' + tx' = 0$, $x(1) = 0$, $x'(1) = 2$.

   h) $t^2x'' - tx' + 2x = 0$, $x(1) = 0$, $x'(1) = 1$.

2. Solve the initial value problem $x'' + t^2x' = 0$, $x(0) = 0$, $x'(0) = 1$. Is this a Cauchy-Euler equation?

3. This exercise presents a method for solving a Cauchy–Euler equation using a change of the independent variable. Show that the transformation $\tau = \ln t$ to a new independent variable τ transforms the Cauchy–Euler equation $at^2x'' + btx' + cx = 0$ into an linear equation with constant coefficients. Use this method to solve Exercise 1a.

4. Find the general solution to the equation $a(t)x'' + a'(t)x' = f(t)$. Your answer should be expressed in terms of integrals.

2.4.2 Variation of Parameters

There is a general formula, called the variation of parameters formula, for the particular solution to a nonhomogeneous linear equation

$$x'' + p(t)x' + q(t)x = f(t). \tag{2.25}$$

It requires knowledge of a pair of fundamental solutions of the homogeneous equation.

The idea is as follows. Let $x_1(t)$ and $x_2(t)$ be independent solutions to the homogeneous equation

$$x'' + p(t)x' + q(t)x = 0.$$

Then

$$x_h(t) = c_1x_1(t) + c_2x_2(t)$$

is the general solution of the homogeneous equation. To find a particular solution we assume that c_1 and c_2 vary as functions of time t, and take and take

$$x_p(t) = c_1(t)x_1(t) + c_2(t)x_2(t), \tag{2.26}$$

where now $c_1(t)$ and $c_2(t)$ are functions to be determined. (Hence the term variation of parameters.) Substituting this expression into the nonhomogeneous equation to obtain expressions for $c_1(t)$ and $c_2(t)$ is a tedious task in calculus and algebra, and we leave most of the details to the reader. Here is how the argument goes. We calculate x_p' and x_p'' and substitute into the equation. For notational simplicity, we drop the t variable in all of the functions. We have

$$x_p' = c_1x_1' + c_2x_2' + c_1'x_1 + c_2'x_2.$$

Let us set

$$c_1'x_1 + c_2'x_2 = 0. \tag{2.27}$$

Then

$$x_p' = c_1x_1' + c_2x_2',$$
$$x_p'' = c_1x_1'' + c_2x_2'' + c_1'x_1' + c_2'x_2'.$$

Substituting these into the nonhomogeneous differential equation gives

$$c_1x_1'' + c_2x_2'' + c_1'x_1' + c_2'x_2' + p(t)[c_1x_1' + c_2x_2'] + q(t)[c_1x_1 + c_2x_2] = f(t).$$

Because x_1 and x_2 both satisfy the homogeneous equation, the last equation simplifies to

$$c_1'x_1' + c_2'x_2' = f(t). \tag{2.28}$$

Equations (2.27) and (2.28) form a system of two linear algebraic equations in the two unknowns c_1' and c_2'. If we solve these equations simultaneously and then integrate, we finally obtain (readers should fill in the details)

$$c_1(t) = -\int \frac{x_2(t)f(t)}{W(t)}dt, \quad c_2(t) = \int \frac{x_1(t)f(t)}{W(t)}dt,$$

where

$$W(t) = x_1(t)x_2'(t) - x_1'(t)x_2(t). \tag{2.29}$$

The expression $W(t)$ is called the **Wronskian**. Combining the previous expressions gives the **variation of parameters formula** for the particular solution of (2.25):

$$x_p(t) = -x_1(t)\int \frac{x_2(t)f(t)}{W(t)}dt + x_2(t)\int \frac{x_1(t)f(t)}{W(t)}dt. \tag{2.30}$$

The general solution of (2.25) is the homogeneous solution $x_h(t)$ plus this particular solution $x_p(t)$. If the antiderivatives in (2.30) cannot be computed explicitly, then the integrals in (2.30) should be written with variable upper limits of integration:

$$x_p(t) = -x_1(t)\int_0^t \frac{x_2(s)f(s)}{W(s)}ds + x_2(t)\int_0^t \frac{x_1(s)f(s)}{W(s)}ds.$$

Finally, we summarize with the following result.

Theorem 2.32

The general solution to the nonhomogeneous equation

$$x'' + p(t)x' + q(t)x = f(t)$$

is

$$x(t) = c_1x_1(t) + c_2x_2(t) - x_1(t)\int_0^t \frac{x_2(s)f(s)}{W(s)}ds + x_2(t)\int_0^t \frac{x_1(s)f(s)}{W(s)}ds,$$

where c_1 and c_2 are arbitrary constants and $x_1(t)$ and $x_2(t)$ are independent solutions to the homogeneous equation. □

Example 2.33

Find a particular solution to the DE

$$x'' + 9x = 3\sec 3t.$$

Notice this forcing term is not of the type applicable for the method of undetermined coefficients. The homogeneous equation $x''+9x = 0$ has two independent solutions, $x_1 = \cos 3t$ and $x_2 = \sin 3t$. The Wronskian is

$$W(t) = 3\cos^2 3t + 3\sin^2 3t = 3.$$

Therefore, using (),

$$c_1(t) = -\int \frac{\sin 3t \cdot 3\sec 3t}{3}dt, \quad c_2(t) = \int \frac{\cos 3t \cdot 3\sec 3t}{3}dt.$$

Simplifying,

$$c_1(t) = -\int \tan 3t dt = \frac{1}{3}\ln(\cos 3t), \quad c_2(t) = \int 1 dt = t.$$

We do not need constants of integration because we seek only the particular solution. Therefore the particular solution is

$$x_p(t) = \frac{1}{3}\ln(\cos 3t) + t\sin 3t.$$

The general solution is therefore

$$x(t) = c_1\cos 3t + c_2\sin 3t + \frac{1}{3}\ln(\cos 3t) + t\sin 3t.$$

The constants are determined by initial data, if given. □

Remark 2.34

If there are homogeneous initial conditions $x(0) = x'(0) = 0$, then the general solution given in Theorem 2.32 implies $c_1 = c_2 = 0$. The solution to the initial value problem

$$x'' + p(t)x' + q(t)x = f(t), \quad x(0) = 0, \ \ x'(0) = 0$$

is therefore

$$x(t) = \int_0^t \frac{-x_1(t)x_2(s) + x_1(s)x_2(t)}{W(s)} f(s)\, dx.$$

If we denote

$$G(t,s) = \frac{-x_1(t)x_2(s) + x_1(s)x_2(t)}{W(s)},$$

then the solution takes the simple form

$$x(t) = \int_0^t G(t,s)f(s)ds. \tag{2.31}$$

The function $G(t,s)$ is called the **causal Green's function**, or the **influence function**. In higher mathematics this approach plays an essential role in understanding the structure of equations. Briefly, the differential equation can be thought of as an *operator equation*

$$Lx(t) = f(t),$$

where

$$L = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$$

is a *differential operator*. Then (2.31) expresses the fact that the solution can be expressed as

$$x(t) = L^{-1}f(t)$$

where L^{-1}, an *integral operator*, is the inverse of the differential operator L. The scope of these ideas is difficult to imagine at this time, but the idea is central in unifying the theory of linear equations of all types. □

In summary, when a second-order equation has constant coefficients and the forcing term is a polynomial, exponential, sine or cosine, or some combination, then the method of undetermined coefficients may work more easily than variation of parameters. Of course, the easiest method of all is to use a computer algebra system. After we have paid our dues by applying analytic methods to many problems, then we have our license and may use a computer algebra system. The variation of parameters formula is important because it is often used in the theoretical analysis of problems in advanced differential equations.

EXERCISES

1. Use the variation of parameters formula to find a particular solution to the following equations.

   a) $x'' + x = \tan t$.
   b) $x'' - x = te^t$.
   c) $x'' - x = \frac{1}{t}$.
   d) $t^2x'' - 2x = t^3$.
   e) $x'' + x = \frac{1}{t+1}$.
   f) $x'' - 2x' + x = \frac{1}{2t}e^t$.

2. Find the general solution of the equation $x'' + \frac{1}{t}x' = a$, where a is a constant.
3. Find the general solution of the equation $t^2x'' - 3tx' + 3x = 4t^7$.
4. Let $x_1 = x_1(t)$ and $x_2 = x_2(t)$ be independent solutions of the linear equation $x'' + p(t)x' + q(t)x = 0$ on an interval I and let $W(t)$ be the Wronskian of x_1 and x_2.
   a) By direct differentiation show that the Wronskian satisfies the differential equation $W'(t) = -p(t)W(t)$.
   b) Solve this equation for $W(t)$ and show that either $W(t) = 0$ for all $t \in I$, or $W(t)$ is never zero on I.
   c) If $t = a$ and $t = b$ are two adjacent zeros of $x_1(t)$, i.e., $x_1(a) = x_1(b) = 0$, show that $x_2(t)$ must have a zero between a and b. Hint: Use part (b).
5. Let $Lx \equiv x'' + px' + cx$, where p and q are constant, be a differential expression. Suppose $x = \phi(t)$ is a solution to $L\phi = 0$, with $\phi(0) = 0$, $\phi'(0) = 1$. Show that a particular solution to $Lx = f(t)$ is given by $x_p(t) = \int_a^t \phi(t-s)f(s)ds$, where a is any constant. Hint: Leibniz rule, Exercise 17 Sec. 1.2.
6. Use the method of the previous problem to find a particular solution to the equation $x'' - 2mx' + m^2x = f(t)$. Hint: Take $\phi(t) = te^{mt}$.
7. Find a particular solution of $x'' - x = e^t/(1 + e^t)$.

2.4.3 Reduction of Order*

If one solution $x_1(t)$ of

$$x'' + p(t)x' + q(t)x = 0$$

is known, then a second, linearly independent solution $x_2(t)$ can be found of the form

$$x_2(t) = v(t)x_1(t)$$

for some $v = v(t)$ to be determined. To determine $v(t)$ we substitute this form for $x_2(t)$ into the differential equation to obtain a second-order equation for $v(t)$ which can be immediately reduced to a first-order equation. The method is called **reduction of order**; we illustrate it with a simple example.

Example 2.35

Consider the equation

$$x'' - \frac{1}{t}x' + \frac{1}{t^2}x = 0.$$

An obvious solution is $x_1(t) = t$. So let $x_2 = v(t)t$. Substituting, we get

$$(2v' + tv'') - \frac{1}{t}(v + tv') + \frac{1}{t^2}vt = 0,$$

which simplifies to

$$tv'' + v' = 0.$$

Letting $w = v'$, we get the first-order equation

$$tw' + w = 0.$$

Separating variables and integrating easily gives $w = 1/t$. Hence $v = \int(1/t)dt = \ln t$, and the second independent solution is $x_2(t) = t\ln t$. Consequently, the general solution of the equation is

$$x(t) = c_1 t + c_2 t\ln t.$$

Note that this example is a Cauchy–Euler equation with equal roots 1, 1 of the indicial equation. It exposes our choice of $t\ln t$ as the second independent solution. □

EXERCISES

1. Find the general solution of $x'' + tx' + x = 0$ given that $x = e^{-t^2/2}$ is one solution.
2. Show that $x_1(t) = t$ is a solution of the equation $x'' - tx' + x = 0$. Use reduction of order to find a second solution $x_2(t)$.

3. **(a)** For what value(s) of β is $x = t^\beta$ a solution to the *Legendre equation* $(1-t^2)x'' - 2tx' + 2x = 0$? **(b)** Find the general solution to this equation. Hint: Use part (a) and reduction of order to show

$$x(t) = 1 - \frac{t}{2} \ln \frac{1+x}{1-x}$$

is a solution.

4. Consider the equation $x'' - 2ax' + a^2 x = 0$, which has solution $x = e^{at}$. Use reduction of order to find a second independent solution. (This shows the origin of the te^{at} solution in a second-order linear equation with constant coefficients, in the real, equal eigenvalue case.)

5. One solution of

$$x'' - \frac{t+2}{t} x' + \frac{t+2}{t^2} x = 0$$

is $x_1(t) = t$. Find a second independent solution.

6. One solution of *Bessel's equation*

$$t^2 x'' + tx' + \left(t^2 - \frac{1}{4}\right) x = 0$$

is $x_1(t) = \cos t/\sqrt{t}$. Find a second independent solution.

7. Let $y(t)$ be one solution of the equation $x'' + p(t)x' + q(t)x = 0$. Show that the reduction of order method with $x(t) = z(t)y(t)$ leads to the first-order linear equation

$$yz' + (2y' + py)z = 0, \quad z = v'.$$

Show that

$$z(t) = \frac{Ce^{-\int p(t)dt}}{y(t)^2},$$

and then find a second linear independent solution of the equation in the form of an integral.

8. Use ideas from the last exercise to find a second-order linear equation that has independent solutions e^t and $\cos t$.

9. Consider the second-order equation

$$x'' + p(t)x' + q(t)x = 0. \tag{2.32}$$

   a) Use the transformation $x = exp\left(\int y(t)dt\right)$ to convert this equation to a **Riccati equation**

$$y' + y^2 + p(t)y + q(t) = 0.$$

b) Conversely, show that the Riccati equation can be reduced to (2.32) using the transformation $y = x'/x$.

c) Solve the first-order equation

$$y' = -y^2 + \frac{3}{t}y.$$

d) Solve the equation

$$y' + ay = -by^2 + c$$

using the transformation $y = x'/bx$.

e) A chemical **C** in a reactor undergoes the reaction 2**C** →products. Solve the chemical reactor equation

$$VC' = q(C_{\text{in}} - C) - kVC^2.$$

2.5 Higher-Order Equations*

So far we have dealt with first- and second-order equations. Higher-order equations occur in some applications. For example, in solid mechanics the vertical deflection $y = y(x)$ of a beam from its equilibrium satisfies a fourth-order equation. However, the applications of higher-order equations are not as extensive as those for their first- and second-order counterparts.

Here, we outline the basic results for a linear, nth-order equation with constant coefficients. We will observe that the method we used for second-order equations extends easily to the higher order case. Consider

$$x^{(n)} + p_{n-1}x^{(n-1)} + \cdots + p_1x' + p_0x = 0, \tag{2.33}$$

where the p_i, $i = 0, 1, ..., n-1$, are specified constants. The *general solution* of (2.33) has the form

$$x(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t),$$

where $x_1(t), x_2(t), ..., x_n(t)$ are independent solutions, and $c_1, c_2, ..., c_n$ are arbitrary constants. Hence, the general solution is a linear combination of n different basic solutions. To find these basic solutions we adopt the same strategy as earlier and assume a solution of the form of an exponential function

$$x(t) = e^{\lambda t},$$

where λ is to be determined. Substituting into the equation gives

$$\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0 = 0, \tag{2.34}$$

which is an nth-degree polynomial equation for λ. Equation (2.34) is the **characteristic equation**. From algebra we know there are n roots $\lambda_1, \lambda_2, ..., \lambda_n$. As before, we call these **eigenvalues**. Here we are counting multiple roots and complex roots, the latter of which occur in complex conjugate pairs $a \pm bi$. A root $\lambda = a$ has *multiplicity* K if $(\lambda - a)^K$ appears in the factorization of the characteristic polynomial.

Of course, the roots for higher order polynomial equations cannot, in general, be found by a formula unless the equation is extremely simple. Therefore, we often resort to computer software to find them.

If the roots are all real and distinct, we obtain n different basic solutions $x_1(t) = e^{\lambda_1 t}$, $x_2(t) = e^{\lambda_2 t}, \ldots, x_n(t) = e^{\lambda_n t}$. In this case the general solution of (2.33) is a linear combination of these,

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_n e^{\lambda_n t}. \tag{2.35}$$

If some roots of (2.34) are not real, then we proceed as expected from the study of second-order equations. A complex conjugate pair, $\lambda = a \pm ib$ gives rise to two real solutions $e^{at} \cos bt$ and $e^{at} \sin bt$. A real, double root λ (multiplicity 2) leads to two solutions $e^{\lambda t}$ and $te^{\lambda t}$. A root λ of multiplicity 3 leads to three independent solutions, $e^{\lambda t}$, $te^{\lambda t}$, $t^2 e^{\lambda t}$, and so on. Proceeding in this way we can build up n basic solutions from the factorization of the characteristic polynomial (2.34).

The general solution of an nth-order nonhomogeneous equation of the form

$$x^{(n)} + p_{n-1} x^{(n-1)} + \cdots + p_1 x' + p_0 x = g(t), \tag{2.36}$$

is the sum of the general solution of the homogeneous equation (2.33) and a particular solution to the equation (2.36). This result is in fact true even if the coefficients p_i are functions of t. For the constant coefficient case, the particular solution can be found using the method of undetermined coefficients in exactly the same way as for second-order equations.

Example 2.36

If the characteristic equation for a sixth-order equation has eigenvalues $\lambda = -2 \pm 3i, 4, 4, 4, -1$, the general solution is

$$x(t) = c_1 e^{-2t} \cos 3t + c_2 e^{-2t} \sin 3t + c_3 e^{4t} + c_4 t e^{4t} + c_5 t^2 e^{4t} + c_6 e^{-t}. \quad \square$$

Example 2.37

Find a differential equation whose basic solutions are e^{3t}, te^{3t}, and e^{-t}. The characteristic roots or eigenvalues are $\lambda = 3$, 3, and -1. So, 3 is a root of multiplicity two. Therefore the characteristic equation must be

$$(\lambda - 3)^2(\lambda + 1) = 0.$$

Expanding, we get

$$\lambda^3 - 5\lambda^2 + 3\lambda + 9 = 0.$$

Therefore the differential equation is

$$x''' - 5x'' + 3x' + 9x = 0. \quad \square$$

Initial conditions for an nth-order equation (2.33) at $t = 0$ take the form

$$x(0) = \alpha_1, \quad x'(0) = \alpha_2, \ldots, x^{(n-1)}(0) = \alpha_{n-1},$$

where the α_i are given constants. Thus, for an nth-order initial value problem we specify the value of the function and all of its derivatives up to the $(n-1)$st-order, at the initial time. These initial conditions determine the n arbitrary constants in the general solution and select a unique solution to the initial value problem.

Example 2.38

Consider the nonhomogeneous

$$x''' - 2x'' - 3x' = 5e^{4t}.$$

The characteristic equation for the homogeneous equation is

$$\lambda^3 - 2\lambda^2 - 3\lambda = 0,$$

or

$$\lambda(\lambda - 3)(\lambda + 1) = 0.$$

The eigenvalues are $\lambda = 0, -1, 3$, and therefore the homogeneous equation has solution

$$x_h(t) = c_1 + c_2e^{-t} + c_3e^{3t}.$$

The particular solution will have the form $x_p(t) = ae^{4t}$. Substituting into the original nonhomogeneous equation gives $a = 1/4$. Therefore the general solution to the equation is

$$x(t) = c_1 + c_2e^{-t} + c_3e^{3t} + \frac{1}{4}e^{4t}.$$

The three constants can now be determined from initial conditions. For example, for a third-order equation the initial conditions at time $t = 0$ have the form

$$x(0) = \alpha, \quad x'(0) = \beta, \quad x''(0) = \gamma,$$

for some given constants α, β, γ. Of course, initial conditions can be prescribed at any other time t_0. $\square$

EXERCISES

1. Find the general solution of the following differential equations.

a) $x''' + x' = 0$.

b) $x'''' + x' = 1$.

c) $x'''' + x'' = 0$.

d) $x''' - x' - 8x = 0$.

e) $x''' + x'' = 2e^t + 3t^2$.

f) $x''' - 8x = 0$.

2. Solve the initial value problem $x''' + x'' - 4x' - 4x = 0$, $x(0) = 1$, $x'(0) = 0$, $x''(0) = -1$. Hint: Guess one eigenvalue.

3. Write down a linear, fourth-order differential equation whose general solution is
$$x(t) = c_1 + c_2 t + e^{5t}(c_4 \cos 2t + c_5 \sin 5t).$$

4. What is the general solution of a fourth-order differential equation if the four eigenvalues are $\lambda = 3 \pm i,\ 3 \pm i$? What is the differential equation?

2.6 Steady–State Heat Conduction*

Let us consider the following problem in steady-state heat conduction. A cylindrical, uniform, metallic bar of length L and cross-sectional area A is insulated on its lateral side. We assume the left face at $x = 0$ is maintained at T_0 degrees and that the right face at $x = L$ is held at T_L degrees. What is the temperature distribution $u = u(x)$ in the bar after it comes to equilibrium? Here $u(x)$ represents the temperature of the entire cross-section of the bar at position x, where $0 < x < L$. We are assuming that heat flows only in the axial direction along the bar, and any transients caused by initial temperatures in the bar have decayed away. In other words, we have waited long enough for the temperature to reach a steady state. One can conjecture that the temperature distribution is a linear function of x along the bar; that is, $u(x) = T_0 + ((T_L - T_0)/L)x$. This is indeed the case, which we show below. But also we want to consider more complicated problems where the bar has both a variable conductivity and an internal heat source along its length. An internal heat source, for example, could be resistive heating produced by a current running through the bar.

The physical law providing the basic model is *conservation of energy*. If $[x, x + dx]$ is any small section of the bar, then the rate that heat flows in at x, minus the rate that heat flows out at $x+dx$, plus the rate that heat is generated

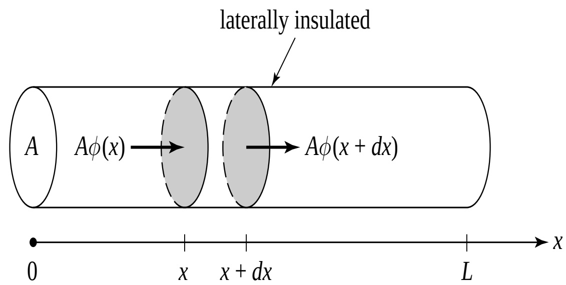

Figure 2.11 Cylindrical bar, laterally insulated, through which heat is flowing in the x-direction. The temperature is uniform in a fixed cross-section at x.

by sources, must equal zero, because the system is in a steady state. Refer to Figure 2.11.

If we denote by $\phi(x)$ the rate that heat flows to the right at any section x (measured in calories/(area · time), and we let $f(x)$ denote the rate that heat is internally produced at x, measured in calories/(volume · time), then

$$A\phi(x) - A\phi(x + dx) + f(x)Adx = 0.$$

Canceling A, dividing by dx, and rearranging gives

$$\frac{\phi(x + dx) - \phi(x)}{dx} = f(x).$$

Taking the limit as $dx \to 0$ yields

$$\phi'(x) = f(x). \tag{2.37}$$

This is an expression of energy conservation in terms of flux. But what about temperature? Empirically, the flux $\phi(x)$ at a section x is found to be proportional to the negative temperature gradient $-u'(x)$ (which measures the steepness of the temperature distribution, or profile, at that point), or

$$\phi(x) = -K(x)u'(x). \tag{2.38}$$

This is **Fourier's heat conduction law**. The given proportionality factor $K(x)$ is called the *thermal conductivity*, in units of energy/(length · degrees · time), which is a measure of how well the bar conducts heat at location x. For a uniform bar K is constant. The minus sign in (2.38) means that heat flows from higher temperatures to lower temperatures. Fourier's law seems intuitively correct and it conforms with the second law of thermodynamics; the larger the temperature gradient, the faster heat flows from high to low temperatures. Combining (2.37) and (2.38) leads to the equation

$$-(K(x)u'(x))' = f(x), \quad 0 < x < L, \tag{2.39}$$

which is the **steady-state heat conduction equation**. When the *boundary conditions*

$$u(0) = T_0, \quad u(L) = T_1, \tag{2.40}$$

are appended to (2.39), we obtain a **boundary value problem** for the temperature $u(x)$. Boundary conditions are conditions imposed on the unknown state u given at different values of the independent variable x, unlike initial conditions that are imposed at a single value. For boundary value problems we usually use x as the independent variable because boundary conditions refer to the boundary of a spatial domain; typically, boundary value problems describe steady-state phenomena where the solution is time-independent.

Note that we can expand the heat conduction equation as

$$-K(x)u''(x) - K'(x)u'(x) = f(x),$$

but there is little advantage in doing so.

Example 2.39

If there are no sources ($f(x) = 0$) and if the thermal conductivity $K(x) = K$ is constant, then the boundary value problem is simply

$$u'' = 0, \quad 0 < x < L,$$
$$u(0) = T_0, \quad u(L) = T_1.$$

Thus the bar is homogeneous and can be characterized by a constant conductivity. The general solution of $u'' = 0$ is $u(x) = c_1 x + c_2$. Applying the boundary conditions determines the constants c_1 and c_2 and gives the linear temperature distribution

$$u(x) = T_0 + \frac{T_L - T_0}{L} x,$$

as previously conjectured. □

In nonuniform systems the thermal conductivity K depends upon location x. As well, K may depend upon the temperature u. Moreover, the heat source term f can depend on location and temperature. In these cases the steady-state heat conduction equation (2.39) takes the more general form

$$-(K(x,u)u')' = f(x,u),$$

which is a nonlinear second-order equation for the steady-state temperature distribution $u = u(x)$.

Boundary conditions at the ends of the bar may specify the flux rather than the temperature. For example, in a homogeneous system, if heat is injected at $x = 0$ at a rate of N calories per area per time, then the left boundary condition takes the form $\phi(0) = N$, or

$$-Ku'(0) = N.$$

This **flux condition** at an endpoint imposes a condition on the derivative of the temperature at that endpoint. In the case that an end, say at $x = L$, is insulated, so that no heat passes through that end, then the boundary condition is

$$u'(L) = 0,$$

which is called an **insulated boundary condition**. As the reader can see, there are myriad interesting boundary value problems associated with heat flow. Similar equations arise in diffusion processes in biology and chemistry, for example, in the diffusion of toxic substances where the unknown is the chemical concentration.

Boundary value problems are much different from initial value problems in that they may have no solution, or they may have infinitely many solutions.

Example 2.40

When $K = 1$ and the heat source term is $f(u) = 9u$ and both ends of a bar of length $L = 2$ are held at $u = 0$ degrees, the boundary value problem becomes

$$\begin{aligned} -u'' &= 9u, \quad 0 < x < 2, \\ u(0) &= 0, \quad u(2) = 0. \end{aligned}$$

The general solution to the DE is $u(x) = c_1 \sin 3x + c_2 \cos 3x$, where c_1 and c_2 are arbitrary constants. Applying the boundary condition at $x = 0$ gives $u(0) = c_1 \sin(3 \cdot 0) + c_2 \cos(3 \cdot 0) = c_2 = 0$. So the solution must have the form $u(x) = c_1 \sin 3x$. Next apply the boundary condition at $x = 2$. Then $u(2) = c_1 \sin(6) = 0$, to obtain $c_1 = 0$. We have shown that the only solution is $u(x) = 0$. There is no nontrivial steady state. But if we make the bar length π, then we obtain the boundary value problem

$$\begin{aligned} -u'' &= 9u, \quad 0 < x < \pi, \\ u(0) &= u(\pi) = 0. \end{aligned}$$

The reader should check that this boundary value problem has infinitely many solutions $u(x) = c_1 \sin 3x$, where c_1 is any number. If we change the right boundary condition, one can check that the boundary value problem

$$-u'' = 9u, \quad 0 < x < \pi,$$
$$u(0) = 0, \quad u(\pi) = 1,$$

has no solution at all. □

Example 2.41

Find all real values of λ for which the boundary value problem

$$-u'' = \lambda u, \quad 0 < x < \pi, \tag{2.41}$$
$$u(0) = 0, \quad u'(\pi) = 0, \tag{2.42}$$

has a nontrivial solution. These values are called the **eigenvalues**, and the corresponding nontrivial solutions are called the **eigenfunctions**. Interpreted in the heat flow context, the left boundary is held at zero degrees and the right end is insulated. The heat source is $f(u) = \lambda u$. We are trying to find which linear heat sources lead to physically meaningful, nontrivial steady states.

To solve this problem we consider different cases because the form of the solution has a different form for $\lambda = 0$, $\lambda < 0$, $\lambda > 0$. If $\lambda = 0$ then the general solution of $u'' = 0$ is $u(x) = ax + b$. Then $u'(x) = a$. The boundary condition $u(0) = 0$ implies $b = 0$ and the boundary condition $u'(\pi) = 0$ implies $a = 0$. Therefore, when $\lambda = 0$, we get only a trivial solution. Next consider the case $\lambda < 0$ so that the general solution has the form

$$u(t) = a \sinh \sqrt{-\lambda} x + b \cosh \sqrt{-\lambda} x.$$

The condition $u(0) = 0$ forces $b = 0$. Then $u'(t) = a\sqrt{-\lambda} \cosh \sqrt{-\lambda} x$. The right boundary condition becomes $u'(\pi) = a\sqrt{-\lambda} \cosh(\sqrt{-\lambda} \cdot 0) = 0$, giving $a = 0$. Recall that $\cosh 0 = 1$. Again there is only the trivial solution. Finally assume $\lambda > 0$. Then the general solution takes the form

$$u(t) = a \sin \sqrt{\lambda} x + b \cos \sqrt{\lambda} x.$$

The boundary condition $u(0) = 0$ forces $b = 0$. Then $u(t) = a \sin \sqrt{\lambda} x$ and $u'(x) = a\sqrt{\lambda} \cos \sqrt{\lambda} x$. Applying the right boundary condition gives

$$u'(\pi) = a\sqrt{\lambda} \cos \sqrt{\lambda} \pi = 0.$$

Now we do not have to choose $a = 0$ (which would again give the trivial solution) because we can satisfy this last condition with

$$\cos \sqrt{\lambda} \pi = 0.$$

The cosine function is zero at the values $\pi/2 \pm n\pi$, $n = 0, 1, 2, 3, \ldots$. Therefore

$$\sqrt{\lambda}\pi = \pi/2 + n\pi, \qquad n = 0, 1, 2, 3, ...$$

Solving for λ yields

$$\lambda = \left(\frac{2n+1}{2}\right)^2, \qquad n = 0, 1, 2, 3, ...$$

Consequently, the values of λ for which the original boundary value problem has a nontrivial solution are $\frac{1}{4}, \frac{9}{4}, \frac{25}{4}, ...$ These are the eigenvalues. The corresponding solutions are

$$u(x) = a \sin\left(\frac{2n+1}{2}\right)x, \quad n = 0, 1, 2, 3, \ldots.$$

These are the eigenfunctions. Notice that the eigenfunctions are unique only up to a constant multiple. In terms of heat flow, the eigenfunctions represent possible steady-state temperature profiles in the bar. The eigenvalues are those values λ for which the boundary value problem has steady-state profiles. □

Boundary value problems are of great interest in applied mathematics, science, and engineering. They arise in many contexts other than heat flow, including wave motion, quantum mechanics, and the solution of partial differential equations.

EXERCISES

1. A homogeneous bar of length 40 cm has its left and right ends held at 30°C and 10°C, respectively. If the temperature in the bar is in steady state, what is the temperature in the cross-section 12 cm from the left end? If the thermal conductivity is K, what is the rate that heat is leaving the bar at its right face?

2. The thermal conductivity of a bar of length $L = 20$ and cross-sectional area $A = 2$ is $K(x) = 1$, and an internal heat source is given by $f(x) = 0.5x(L - x)$. If both ends of the bar are maintained at zero degrees, what is the steady-state temperature distribution in the bar? Sketch a graph of $u(x)$. What is the rate that heat is leaving the bar at $x = 20$?

3. For a metal bar of length L with no heat source and thermal conductivity $K(x)$, show that the steady temperature in the bar has the form

$$u(x) = c_1 \int_0^x \frac{dy}{K(y)} + c_2,$$

where c_1 and c_2 are constants. What is the temperature distribution if both ends of the bar are held at zero degrees? Find an analytic formula and plot the temperature distribution in the case that $K(x) = 1 + x$. If the left end is held at zero degrees and the right end is insulated, find the temperature distribution and plot it.

4. Determine the values of λ for which the boundary value problem

$$-u'' = \lambda u, \quad 0 < x < 1,$$
$$u(0) = u(1) = 0,$$

has a nontrivial solution.

5. Consider the nonlinear heat flow problem

$$(uu')' = 0, \quad 0 < x < \pi,$$
$$u(0) = 0, \quad u'(\pi) = 1,$$

where the thermal conductivity depends on temperature and is given by $K(u) = u$. Find the steady-state temperature distribution.

6. Find all values of λ for which the boundary value problem

$$-u'' - 2u' = \lambda u, \quad 0 < x < 1,$$
$$u(0) = 0, \quad u(1) = 0,$$

has a nontrivial solution.

7. Show that the eigenvalues of the boundary value problem

$$-u'' = \lambda u, \quad 0 < x < 1,$$
$$u'(0) = 0, \quad u(1) + u'(1) = 0,$$

are given by the numbers $\lambda_n = p_n^2$, $n = 1, 2, 3, ...$, where the p_n are roots of the equation $\tan p = 1/p$. Plot graphs of $\tan p$ and $1/p$ and indicate graphically the locations of the values p_n. Numerically calculate the first four eigenvalues.

8. Find the values of λ (eigenvalues) for which the boundary value problem

$$-x^2 u'' - xu' = \lambda u, \quad 1 < x < e^{\pi},$$
$$u(1) = 0, \quad u(e^{\pi}) = 0,$$

has a nontrivial solution. Hint: This is a Cauchy–Euler equation.

3
Laplace Transforms

In this chapter we introduce a dramatically different technique, called the Laplace transform method, for solving the linear, nonhomogeneous equation

$$ax'' + bx' + cx = f(t), \quad x(0) = x_0,\ x'(0) = x_1,$$

where a, b, and c are constant coefficients. The method is based upon transforming the differential equation into an algebraic equation, and it is especially applicable to equations containing a nonhomogeneous forcing term $f(t)$ that is either *discontinuous* or a force that is applied only at a single instant of time (an *impulse*). The methods introduced in Chapter 2, undetermined coefficients and variation of parameters, are extremely inadequate in these physically important cases. Moreover, transform methods are one of the most important tools in all of mathematics and the pure and applied sciences. They are applicable to ordinary and partial differential equations, integral equations, and they are a key component in understanding the stability characteristics of feedback control systems. These are standard methods for engineers, physicists, and applied mathematicians.

The transform goes back to the late 1700s and is named for the great French mathematician and scientist Pierre de Laplace, although some of the basic definitions go back earlier to Leonard Euler. The English engineer Oliver Heaviside developed much of the operational calculus for transform methods in the early 1900s.

The material in this chapter is independent from the remaining chapters, so it may be read at any time.

3.1 Definition and Basic Properties

A successful strategy in many problems is to transform them into simpler ones that can be solved more easily. Many differential equations can can be handled in this way using *integral transform methods*. The Laplace transform, which is one of several transforms used in differential equations, has the effect of turning a differential equation with state function $x = x(t)$ into an algebraic equation for an associated transformed function $X = X(s)$; we can easily solve for $X(s)$ and then return to $x(t)$ via an inverse transformation.

Definition 3.1

Let $x = x(t)$ be a given function defined on the interval $0 \leq t < \infty$. The **Laplace transform** of $x(t)$ is the function $X(s)$ defined by

$$X(s) = \int_0^\infty x(t)e^{-st}dt, \tag{3.1}$$

provided the improper integral exists, meaning

$$\lim_{b\to\infty} \int_0^b x(t)e^{-st}dt \quad \text{exists.} \quad \Box$$

The integrand in (3.1) is a function of t and s, and we integrate on t, evaluating at $t = 0$ and $t = +\infty$. This leaves a function of s. Often we represent the Laplace transform in function notation,

$$\mathcal{L}[x(t)](s) = X(s) \quad \text{or just} \quad \mathcal{L}[x] = X(s).$$

$\mathcal{L}$ represents a function-like operation, called an operator or transform, whose domain and range are sets of functions; $\mathcal{L}$ takes a function $x(t)$ and transforms it into a new function $X(s)$ (see Figure 3.1). In the context of Laplace transformations, t and x are called the **time domain** variables, and s and X are called the **transform domain** variables. In some applications, s may be complex variable having the form $s = \sigma + i\tau$. Typically, we use lower case letters for time domain functions and upper case letters for transform domain functions.

An important observation is the linearity of the transform.

Theorem 3.2

(**Linearity**) The Laplace transform is a **linear operation**; that is, the Laplace transform of a sum of two functions is the sum of the Laplace transforms of

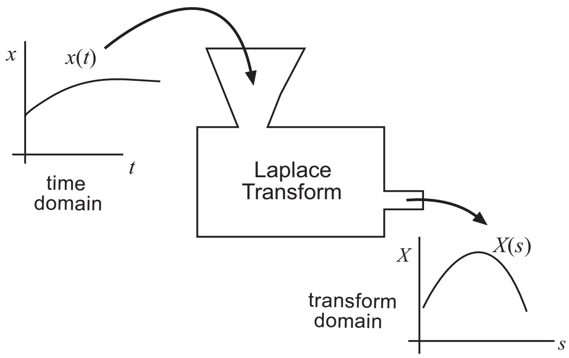

Figure 3.1 The Laplace transform $\mathcal{L}$ as a machine that transforms functions $x(t)$ in the time domain to functions $X(s)$ in the transform domain.

each, and the Laplace transform of a constant times a function is the constant times the transform of the function. We can express these rules in symbols by

$$\mathcal{L}[\alpha x + \beta y] = \alpha\mathcal{L}[x] + \beta\mathcal{L}[y]. \tag{3.2}$$

where $x = x(t)$ and $y = y(t)$ are functions and α and β are any constants. □

Observe that (3.2) is the same as

$$\int_0^\infty (\alpha x(t) + \beta y(t))e^{-st}dt = \alpha\int_0^\infty x(t)e^{-st}dt + +\beta\int_0^\infty y(t)e^{-st}dt,$$

which is just a statement of the properties of an integral.

We can compute the Laplace transform of many common functions directly from the definition (3.1).

Example 3.3

Let $x(t) = e^{at}$. Then

$$\begin{aligned} X(s) &= \lim_{b\to\infty}\int_0^b e^{at}e^{-st}dt = \lim_{b\to\infty}\int_0^b e^{(a-s)t}dt \\ &= \lim_{b\to\infty}\frac{1}{a-s}e^{(a-s)t}\Big|_{t=0}^{t=b} = \lim_{b\to\infty}\frac{1}{s-a}\left(e^{(a-s)b}-1\right). \end{aligned}$$

The limit on the right exists only if $a - s < 0$, or $s > a$. Therefore

$$X(s) = \frac{1}{s-a}, \quad s > a,$$

or, in different notation,

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}, \quad s > a.$$

Notice that there is a on the domain of the transform variable, which is common. We often make the preceding calculation more concise by only writing

$$\begin{aligned} X(s) &= \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{(a-s)t}dt \\ &= \frac{1}{a-s}e^{(a-s)t} \mid_{t=0}^{t=\infty} = \frac{1}{s-a}, \quad s > a. \quad \square \end{aligned}$$

Example 3.4

Let $x(t) = 1$. Then

$$X(s) = \int_0^\infty 1 \cdot e^{-st}dt = \frac{1}{-s}e^{-st}|_{t=0}^{t=\infty} = \frac{1}{s}, \quad s > 0.$$

In different notation, $\mathcal{L}[1] = 1/s$. This transform exists only for $s > 0$; otherwise the improper integral does not converge. $\square$

Example 3.5

Let $x(t) = t$. Then, using integration by parts ($u = t,\ dv = e^{-st}dt$),

$$X(s) = \int_0^\infty te^{-st}dt = \left[t\frac{e^{-st}}{-s}\right]_{t=0}^{t=\infty} - \frac{1}{s}\int_0^\infty 1 \cdot e^{-st}dt = \frac{1}{s^2}, \quad s > 0. \quad \square$$

You may have concluded that calculating Laplace transforms can be tedious business. Fortunately, generations of mathematicians, scientists, and engineers have computed the Laplace transforms of many, many functions, and the results have been catalogued in tables and in software systems. Some of the tables are extensive. In this chapter we require only a short table, which is given at the end of the chapter. The table lists functions $x(t)$ in the first column, and their transforms $X(s)$, or $\mathcal{L}[x]$, in the second. The strategy in this text is to make liberal use of the table and not delve into detailed calculations of transforms.

The Inverse Transform

Given $x(t)$, the Laplace transform $X(s)$ is computed from the definition given in formula (3.1). This suggests the opposite problem: given $X(s)$, find a function $x(t)$ whose Laplace transform is $X(s)$. This is the *inverse problem.*

Unfortunately, there is no elementary formula[1] that gives $x(t)$ in terms of $X(s)$. In general, we use the notation

$$x(t) = \mathcal{L}^{-1}\left[X(s)\right], \quad \text{or} x = \mathcal{L}^{-1}\left[X\right].$$

We call $\mathcal{L}^{-1}$ as the **inverse transform** of x. The functions $x(t)$ and $X(s)$ form a transform pair, and they are listed together in two columns of the table; the left column is the inverse transform of the corresponding element in the right column.

The inverse Laplace transform is a linear operation as well:

$$\mathcal{L}^{-1}[\alpha X(s) + \beta Y(s)] = \alpha\mathcal{L}^{-1}[X(s)] + \beta\mathcal{L}^{-1}[Y(s)]. \tag{3.3}$$

Example 3.6

If $X(s) = 1/(s-a)$, then Example 3.3 shows $x(t) = e^{at}$. Therefore, $x(t) = e^{at}$ is the inverse transform of $X(s) = 1/(s-a)$, and we write

$$e^{at} = \mathcal{L}^{-1}\left[\frac{1}{s-a}\right].$$

Similarly, from Example 3.4

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1,$$

and from Example 3.5

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t. \quad \square$$

As we observed, the Laplace transform is an improper integral. You may ask which functions have Laplace transforms. Clearly, if a function $f(t)$ grows too quickly as t gets large, then the improper integral will not exist and there will be no transform. There are two conditions that guarantee existence, and these are reasonable conditions for most problems in science and engineering. First, we require that $f(t)$ not grow too fast; a way of stating this mathematically is to require that there are constants $M > 0$ and r for which

$$|f(t)| \leq Me^{rt}$$

is valid for all $t > t_0$, where t_0 is some value of time. That is, beyond the value t_0 the function is bounded above and below by an exponential function. Such

[1] The formula for the inverse transform is a contour integral in the complex plane, and is beyond the scope of this text. See Churchill (1958), for example, for a readable treatment.

functions f are said to be of **exponential order**. Second, we require that $f(t)$ be **piecewise continuous** (PWC) on $0 \le t < \infty$. In other words, on any bounded subinterval of $0 \le t < \infty$ we assume that $f(t)$ has at most a finite number of simple discontinuities, and at any point of discontinuity $f(t)$ has finite left and right limits. One can prove that if $f(t)$ is piecewise continuous on $0 \le t < \infty$ and of exponential order, then the Laplace transform $F(s)$ exists for all $s > r$. Even more can be stated. If $f(t)$ is PWC and exponential order, then

$$F(s) \le \frac{K}{s}, \quad K > 0,$$

and $\lim_{s\to\infty} F(s) = 0$. If the limit of a function $G(s)$ does not go to zero as $s \to \infty$, then $G(s)$ cannot be a Laplace transform of some function $g(t)$.

Example 3.7

(**Unit switching function**) As mentioned earlier, discontinuous functions are common forcing functions in physical systems. Now we introduce a simple, unit step function that greatly aids the representation of such functions. We define the **Heaviside function** $H(t)$ by[2]

$$H(t) = \begin{cases} 0, & t < 0; \\ 1, & t \ge 0. \end{cases}$$

Its translation to the right a units is therefore $H(t-a)$, which is defined by

$$H(t-a) = \begin{cases} 0, & t < a; \\ 1, & t \ge a. \end{cases}$$

This function is like a *switch* which is off (zero) if $t < a$, and on (1) when $t \ge a$. In in an circuit, for example, this would mean the switch is *open* for $t < a$ and *closed* for $t > a$. The Laplace transform of $H(t-a)$ is easily calculated by breaking up the integral into two pieces, $t < a$ and $t > a$:

$$\begin{aligned} \mathcal{L}[H(t-a)] &= \int_0^\infty H(t-a)(t)e^{-st}dt \\ &= \int_0^a H(t-a)e^{-st}dt + \int_a^\infty H(t-a)(t)e^{-st}dt \\ &= \int_0^a 0 \cdot e^{-st}dt + \int_a^\infty 1 \cdot e^{-st}dt \\ &= -\frac{1}{s}e^{-st}\,\Big|_{t=a}^{t=\infty} = \frac{1}{s}e^{-as}, \quad s > 0. \end{aligned}$$

[2] The Heaviside function is commonly denoted by H and is the only function we use that breaks the rule of using lower case letters for time domain functions.

Therefore,

$$\mathcal{L}[H(t-a)] = \frac{1}{s}e^{-as}, \quad s > 0.$$

This transform automatically yields the inverse transform

$$H(t-a) = \mathcal{L}^{-1}\left(\frac{1}{s}e^{-as}\right). \quad \square$$

The Heaviside function is useful for expressing multi-lined functions in a single formula.

Example 3.8

The Heaviside function is useful for expressing multi-lined functions in a single formula. For example, consider the step function

$$f(t) = \begin{cases} 3, & 0 \le t < 2 \\ 4, & 2 \le t \le 3 \\ 2, & 3 < t \le 6 \\ 0, & t > 6 \end{cases}$$

This can be written in one line as

$$f(t) = 3H(t) + (4-3)H(t-2)(t) + (2-4)H(t-3)(t) + (0-2)H(t-6)(t).$$

The first term switches on the function 3 at $t = 0$; the second term switches off 3 and switches on 4 at time $t = 2$; the third term switches off 4 and switches on 2 at $t = 3$; finally, the last term switches off 2 at $t = 6$. Using the linearity of the Laplace transform we have

$$F(s) = \mathcal{L}[f(t)] = \frac{3}{s} + \frac{1}{s}e^{-2s} - \frac{2}{s}e^{-3s} - \frac{2}{s}e^{-6s}. \square$$

Two additional operational formulas are extremely useful for solving differential equations.

(a) (**Shift Property**) The Laplace transform of a function times an exponential, $f(t)e^{at}$, is given by,

$$\mathcal{L}[f(t)e^{at}] = F(s-a). \tag{3.4}$$

Multiplying a function $f(t)$ by an exponential has the effect of shifting the transform $F(s)$ of $f(t)$ to $F(s-a)$.

(b) (**Switching Property**) The Laplace transform of a function $f(t)$ that switches *on* at $t = a$ is given by

$$\mathcal{L}[H(t-a)f(t-a)] = e^{-as}F(s). \tag{3.5}$$

This result is often used in its inverse form:

$$\mathcal{L}^{-1}[e^{-as}F(s)] = H(t-a)f(t-a).$$

Proofs of the properties (a) and (b) follow directly from the definition of the Laplace transform, and they are requested in the exercises.

Example 3.9

Find the Laplace transform of the function

$$f(t) = te^{-2t}.$$

We know that the transform of $f(t) = t$ is $F(s) = 1/s^2$. By the shift property,

$$\mathcal{L}[te^{-2t}] = F(s-(-2)) = \frac{1}{(s+2)^2}. \quad \square$$

Example 3.10

Find the inverse transform of the function

$$F(s) = \frac{1}{s-2}e^{-3s}.$$

The exponential on the right signals the switching property. We know from the table that

$$\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = e^{2s}.$$

By the switching property,

$$\mathcal{L}^{-1}\left[\frac{1}{s-2}e^{-3s}\right] = H(t-3)e^{2(s-3)}. \quad \square$$

The exercises contain many examples of illustrating the preceding concepts and extending the Laplace transformation to additional functions.

EXERCISES

1. Use the definition of the Laplace transform to compute the transform of the square pulse function $x(t) = 1$, $1 \le t \le 2$; $x(t) = 0$, otherwise. Plot $x(t)$ on $t \ge 0$, and plot its transform $X(s)$.

2. Use the definition of the Laplace transform to find the transform of $x(t) = e^{-3t}H(t-2)$.

3. Find the Laplace transform of $x(t) = \sin kt$ and $x(t) = \cos kt$ using the definitions

$$\sin kt = \frac{1}{2i}\left(e^{ikt} - e^{-ikt}\right), \quad \cos kt = \frac{1}{2}\left(e^{ikt} + e^{-ikt}\right).$$

Check your answers in the table.

4. Find the Laplace transform of the hyperbolic functions $x(t) = \sinh kt$ and $x(t) = \cosh kt$ using the definitions

$$\sinh kt = \frac{1}{2}\left(e^{kt} - e^{-kt}\right), \quad \cosh kt = \frac{1}{2}\left(e^{kt} + e^{-kt}\right).$$

Check your answers in the table.

5. Derive the operational formulas (3.4) and (3.5) directly from the definition. Hint: Change variables in the integrals.

6. Use the definition of Laplace transform to show that

$$\mathcal{L}\left[f(t)H(t-a)\right] = e^{-as}\mathcal{L}[f(t+a)].$$

7. Use the preceding exercise to compute $\mathcal{L}[t^2 H(t-1)]$.

8. Find the Laplace transform of the following functions.

a) $6 + 5e^{-2t} + te^{3t}$.

b) $tH(t-3)$.

c) $\cos 5t$.

d) $\sin(2t+\pi)$.

e) $3e^{-t}\cosh t$.

f) $H(t-\pi)\cos(t-\pi)$.

9. Find the inverse transform of the following functions.

a) $\frac{7}{s+2}$.

b) $\frac{3}{s} - \frac{2}{s^2+6}$.

c) $\frac{2}{(s-5)^2}$.

d) $\frac{7}{s}e^{-4s}$.

e) $\frac{1}{s(s-2)}e^{-s}$.

f) $\frac{7s+1}{s^2+4}$.

g) $\frac{3}{2s^2+7}$.

h) $e^{-\pi s}\frac{3s}{s^2+9}$

i) $\frac{5s}{(s-3)^2+4}$.

j) $e^{-2s}\frac{3}{s^2}$.

10. Sketch the graphs of $\sin t$, $\sin(t-\pi/2)$, and $H(t-\pi/2)\sin(t-\pi/2)$, and find the Laplace transform of each.

11. Use the shift property to find the Laplace transform of $e^{at}\sin kt$.

12. Use the switching property to find the Laplace transform of

$$x(t) = \begin{cases} 0 & t < 2 \\ e^{-t}, & t > 2. \end{cases}$$

13. Show that

$$\mathcal{L}\left[f(at)\right] = \frac{1}{a}F\left(\frac{s}{a}\right), \ a > 0.$$

14. **(a)** Does the function $x(t) = e^{t^2}$ have a Laplace transform? **(b)** What about $x(t) = 1/t$? Explain why or why not. **(c)** State why $X(s) = \frac{1}{s}e^s$ cannot be the Laplace transform of some function $x = x(t)$.

15. Plot the *square-wave* function

$$f(t) = \sum_{n=0}^{\infty}(-1)^n H(t-n)$$

on the interval $t \geq 0$ and find its transform $F(s)$. Hint: Use the geometric series $1 + z + z^2 + \cdots = 1/(1-z)$ to find the sum.

16. From the definition of the Laplace transform, find $\mathcal{L}\left[1/\sqrt{t}\right]$ using the integral substitution $st = r^2$ and using $\int_0^\infty \exp(-r^2)dr = \sqrt{\pi}/2$.

17. The **Gamma function** is a special function defined by

$$\Gamma(y) = \int_0^\infty e^{-t}t^{y-1}dt, \quad y > -1.$$

It is important in probability, statistics, and many other areas of mathematics, science, and engineering.

a) Show that $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(n+1) = n!$ for nonnegative integers n. Hint: Integrate by parts.

b) Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

c) Show that

$$\mathcal{L}\left[t^a\right] = \frac{\Gamma(a+1)}{s^{a+1}}, \quad s > 0.$$

18. Prove that the inverse transform is a linear operation, that is, it satisfies (3.3).

3.2 Differential Equations

We derived some of the basic properties of Laplace transforms, and now we are prepared to take up the key issue of this chapter, that is, solving differential equations. What makes the Laplace transform so useful for differential equations is this:

> *The Laplace transform turns derivative operations in the time domain into multiplication operations in the transform domain.*

We calculate the transforms of derivatives using the basic integration by parts formula:

$$\int_a^b x(t)y'(t)dt = x(t)y(t)|_a^b - \int_a^b x'(t)y(t)dt.$$

In elementary courses integration by parts is a technique used to calculate integrals, but it is an essential theoretical tool used in differential equations. We think of the formula as a way of removing the derivative on one factor in an integrand and putting it on the other factor, while generating a boundary term. The following theorem gives the fundamental operational formulas for solving differential equations.

Theorem 3.11

Let $x(t)$ be a function and $X(s)$ its Laplace transform. Then

$$\mathcal{L}[x'] = sX(s) - x(0), \tag{3.6}$$

$$\mathcal{L}[x''] = s^2X(s) - sx(0) - x'(0). \quad \Box \tag{3.7}$$

We prove (3.6). Using integration by parts,

$$\begin{aligned}\mathcal{L}[x'] &= \int_0^\infty x'(t)e^{-st}dt = \left[x(t)e^{-st}\right]_{t=0}^{t=\infty} - \int_0^\infty -sx(t)e^{-st}dt \\ &= -x(0) + sX(s), \quad s > 0.\end{aligned}$$

The second operational formula (3.7) can be derived using two successive integrations by parts, and we leave that calculation as an exercise. A simpler argument is to obtain (3.7) using (3.6). Simply,

$$\begin{aligned}\mathcal{L}[x''] &= \mathcal{L}[(x')'] \\ &= s\mathcal{L}[x'] - x'(0) \\ &= s\left(sX(s) - x(0)\right) - x'(0). \qquad \Box\end{aligned}$$

There are similar formulas for transforms of higher derivatives. See Table 3.1.

Example 3.12

As the previous calculation illustrates and suggests, the derivative formulas (3.6)–(3.7) are useful to find transforms without resorting to the integral definition of the Laplace transform. For example, we calculate $\mathcal{L}[t^2]$. First notice that $(t^2)' = 2t$. Then, we have

$$\mathcal{L}[(t^2)'] = s\mathcal{L}[t^2] - 0^2 = \mathcal{L}[2t] = \frac{2}{s^2}.$$

Therefore,

$$\mathcal{L}[t^2] = \frac{2}{s^3}.$$

We can continue this process to compute the transforms of t^3, t^4, $t^5, \ldots$. By induction it follows that

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \quad n = 0, 1, 2, 3, \ldots. \quad \Box$$

Example 3.13

Here is an example using the second derivative formula (3.7). We find $\mathcal{L}[\cosh t]$ by writing

$$\begin{aligned} \mathcal{L}[\cosh t] &= \mathcal{L}[(\cosh t)''] = s^2\mathcal{L}[\cosh t] - s\cosh 0 - \sinh 0 \\ &= s^2\mathcal{L}[\cosh t] - s. \end{aligned}$$

Solving for $\mathcal{L}[\cosh t]$ gives us the transform

$$\mathcal{L}[\cosh t] = \frac{s}{s^2 - 1}. \qquad \Box$$

3.2.1 Initial Value Problems

The derivative formulas (3.6)–(3.7) allow us to transform a differential equation with unknown $x(t)$ into an algebraic problem with unknown $X(s)$. We solve for $X(s)$ and then find $x(t)$ using the inverse transform $x = \mathcal{L}^{-1}[X]$.

The following examples illustrate how this procedure works to solve initial value problems for linear differential equations with constant coefficients. Actually, it is applicable on equations of all orders and on systems of several equations in several unknowns. We assume $x(t)$ is the unknown state function. The idea is to take the transform of each term in the equation, using the linearity property. Then, using Theorem 3.11, reduce all of the derivative terms to algebraic expressions and solve for the transformed state function $X(s)$. Finally,

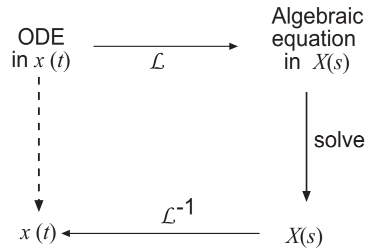

Figure 3.2 An ODE for an unknown function $x(t)$ is transformed to an algebraic equation for its transform $X(s)$. The algebraic problem is solved for $X(s)$ in the transform domain, and the solution is returned to the original time domain via the inverse transform.

invert $X(s)$ to recover the solution $x(t)$. Figure 3.2 illustrates this three-step method. The last step in this procedure, finding the inverse transform, is the most difficult, and in this section we get additional practice in finding inverse transforms.

Example 3.14

Consider the second-order initial value problem

$$x'' + k^2 x = 0, \quad x(0) = 0, \quad x'(0) = 1.$$

Taking transforms of both sides and using the linearity property gives

$$\mathcal{L}[x''] + k^2 \mathcal{L}[x] = \mathcal{L}[0].$$

Then Theorem 3.11 gives

$$s^2 X(s) - sx(0) - x'(0) + k^2 X(s) = 0,$$

which is an *algebraic* equation for the transformed state $X(s)$. Using the initial conditions, we get

$$s^2 X(s) - 1 + k^2 X(s) = 0.$$

Note that the initial conditions are actually part of the transform formula. Solving for the transform function $X(s)$ gives

$$X(s) = \frac{1}{k^2 + s^2} = \frac{1}{k}\frac{k}{s^2 + k^2},$$

which is the solution in the transform domain. Therefore, from the table, the inverse transform is

$$x(t) = \frac{1}{k}\sin kt,$$

which is the solution to the initial value problem. □

Example 3.15

Solve the first-order nonhomogeneous equation

$$x' + 2x = e^{-t}, \quad x(0) = 0.$$

Taking Laplace transforms of each term

$$\mathcal{L}[x'] + \mathcal{L}[2x] = \mathcal{L}[e^{-t}],$$

or

$$sX(s) - x(0) + 2X(s) = \frac{1}{s+1}.$$

Use the initial condition to set $x(0) = 0$ and then solve for the transformed function $X(s)$ to get

$$X(s) = \frac{1}{(s+1)(s+2)}.$$

Now we can look up the inverse transform in the table. We find

$$x(t) = \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = e^{-t} - e^{-2t}. \quad \square$$

Sometimes a table of transforms may not include the exact entry for an inverse transform and we have to algebraically manipulate or simplify it to obtain a table entry. A common technique is to expand complex fractions into their *partial fraction decomposition*, a technique encountered in elementary algebra and in calculus. We observe, however, that there are extensive tables and computer algebra systems containing large numbers of inverse transforms. Thus, the partial fractions technique for inversion is not used as often as in the past, and we illustrate only a few simple examples of the technique.

Example 3.16

(**Partial Fractions, I**) In the last example we obtained

$$X(s) = \frac{1}{(s+1)(s+2)}.$$

We can decompose $X(s)$ as

$$\frac{1}{(s+1)(s+2)} = \frac{a}{(s+1)} + \frac{b}{(s+2)},$$

for some constants a and b to be determined. Combining terms on the right side gives

$$\begin{aligned}\frac{1}{(s+1)(s+2)} &= \frac{a(s+2)+b(s+1)}{(s+1)(s+2)}\\ &= \frac{(a+b)s+2a+b}{(s+1)(s+2)}.\end{aligned}$$

Comparing numerators on the left and right force $a+b=0$ and $2a+b=1$. Hence $a=-b=1$ and we have

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{(s+1)} + \frac{-1}{(s+2)}.$$

We reduced the complex fraction to the sum of two easily identifiable fractions that are found in the table. Using the linearity property of the inverse transform,

$$\begin{aligned}\mathcal{L}^{-1}[X(s)] &= \mathcal{L}^{-1}\left[\frac{1}{(s+1)}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s+2)}\right]\\ &= e^{-t} - e^{-2t}. \qquad \square\end{aligned}$$

Example 3.17

(**Partial Fractions, II**) A common expression encountered in solving differential equations is

$$X(s) = \frac{1}{s^2+bs+c}.$$

If the denominator has two distinct real roots, then it factors and we can proceed as in the previous example. If the denominator has complex roots, then we *complete the square* of the denominator. For example, consider

$$X(s) = \frac{1}{s^2+3s+6}.$$

Then, completing the square in the denominator,

$$\begin{aligned}X(s) &= \frac{1}{s^2+3s+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+6}\\ &= \frac{1}{\left(s+\frac{3}{2}\right)^2+\left(\frac{\sqrt{15}}{2}\right)^2}.\end{aligned}$$

This entry is in the table, up to a factor of $\sqrt{15}/2$. Therefore we multiply and divide by this factor and locate the inverse transform in the table as

$$x(t) = \frac{2}{\sqrt{15}}e^{-3t/2}\sin\frac{\sqrt{15}}{2}t. \qquad \square$$

Piecewise continuous sources

Laplace transform methods are applicable on problems of the form

$$x'' + bx' + cx = f(t), \quad t > 0$$
$$x(0) = x_0, \; x'(0) = x_1.$$

If the forcing function f is a continuous function, then we can use variation of parameters to find the particular solution; if f has the special form of a polynomial, exponential, sine, or cosine, or sums and products of these forms, we can use the method of undetermined coefficients to find the particular solution. If, however, f is a piecewise continuous source with different forms on different intervals, then finding the general solution would be difficult. On each subinterval interval we would have to find the general solution and then determine the arbitrary constants by matching up solutions at the endpoints of the adjacent subintervals. This is an algebraically difficult and tedious task. Using Laplace transforms, however, is not so tedious. Now we present additional examples on how to deal with discontinuous forcing functions.

Example 3.18

As noted earlier, the Heaviside function is used to express piecewise, or multilined, functions in a single line. For example,

$$\begin{aligned} f(t) &= \begin{cases} t, & 0 < t < 1 \\ 2, & 1 \le t \le 3 \\ 0, & t > 3 \end{cases} \\ &= t + (2-t)H(t-1) - 2H(t-3). \end{aligned}$$

The first term switches on the function t at $t = 0$; the second term switches on the function 2 and switches off the function t at $t = 1$; and the last term switches off the function 2 at $t = 3$. By linearity, the Laplace transform of $f(t)$ is given by

$$F(s) = \mathcal{L}[t] + 2\mathcal{L}[H(t-1)] - \mathcal{L}[tH(t-1)] - 2\mathcal{L}[H(t-3)].$$

The first, second, and fourth terms are straightforward:

$$\mathcal{L}[t] = 1/s^2, \;\; \mathcal{L}[H(t-1)] = \frac{1}{s}e^{-s}, \;\; \mathcal{L}[H(t-3)] = \frac{1}{s}e^{-3s.}$$

The third term can be calculated using $\mathcal{L}\left[f(t)H(t-a)\right] = e^{-as}\mathcal{L}[f(t+a)]$. See the table. With $f(t) = t$ we have

$$\mathcal{L}\left[tH(t-1)\right] = e^{-s}\mathcal{L}[t+1] = \frac{1}{s^2}e^{-s} + \frac{1}{s}e^{-s}.$$

Putting all these results together gives

$$F(s) = \frac{1}{s^2} + \frac{2}{s}e^{-s} - \left(\frac{1}{s^2}e^{-s} + \frac{1}{s}e^{-s}\right) - \frac{2}{s}e^{-3s}. \quad \square$$

Example 3.19

Solve the initial value problem

$$x'' + 4x = \sin t - H(t-\pi)\sin t, \quad x(0) = x'(0) = 0.$$

Physically, this equation models an oscillator where the forcing term is one hump of the sine curve on $0 \le t \le \pi$, which then switches off. The Laplace transform of of the equation gives

$$\begin{aligned} s^2 X(s) + 4X(s) &= \mathcal{L}[\sin t - H(t-\pi)\sin t] \\ &= \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1}. \end{aligned}$$

To get the second term on the right we used the table entry $\mathcal{L}[H(t-a)f(t)] = e^{-as}\mathcal{L}[f(t+a]$ with $f(t) = \sin t$. Thus,

$$\begin{aligned} \mathcal{L}[H(t-\pi)\sin t] &= e^{-\pi s}\mathcal{L}[\sin(t+\pi)] \\ &= e^{-\pi s}\mathcal{L}[\sin t \cos \pi + \sin \pi \cos t] \\ &= e^{-\pi s}\mathcal{L}[-\sin t] \\ &= -e^{-\pi s}\frac{1}{s^2+1}. \end{aligned}$$

Therefore,

$$X(s) = \frac{1}{(s^2+1)(s^2+4)} + e^{-\pi s}\frac{1}{(s^2+1)(s^2+4)}.$$

Now we face inversion. Using partial fractions we get

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{A}{s^2+1} + \frac{B}{s^2+4} = \frac{(A+B)s^2 + 4A + B}{(s^2+1)(s^2+4)}.$$

Thus $A + B = 0$ and $4A + B = 1$, giving $A = \frac{1}{3},\ B = -\frac{1}{3}$. Hence,

$$X(s) = \frac{1/3}{s^2+1} - \frac{1/3}{s^2+4} + e^{-\pi s}\left(\frac{1/3}{s^2+1} - \frac{1/3}{s^2+4}\right).$$

Clearly

$$\mathcal{L}^{-1}\left[\frac{1/3}{s^2+1} - \frac{1/3}{s^2+4}\right] = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t.$$

We use the switching property to invert the exponential term. We have

$$\mathcal{L}^{-1}\left[e^{-\pi s}\frac{1}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}H(t-\pi)\sin(t-\pi) - \frac{1}{6}H(t-\pi)\sin 2(t-\pi).$$

Finally, the solution is therefore

$$x(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t + \frac{1}{3}H(t-\pi)\sin(t-\pi)t - \frac{1}{6}H(t-\pi)\sin 2(t-\pi)t. \quad (3.8)$$

A plot of the solution is shown in Figure 3.3. □

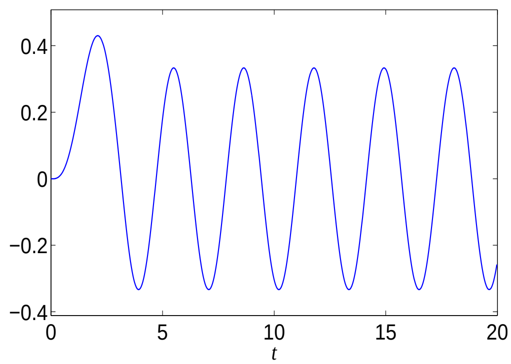

Figure 3.3 The solution (3.8).

Remark 3.20

(**Piecewise continuous inputs**) When the input of a differential equation is only piecewise continuous (PWC), it is not obvious what properties the solution possesses. Consider the differential equation

$$ax'' + bx' + cx = f(t)$$

where f is PWC for $t \geq 0$. Is $x(t)$ continuous? Differentiable? The answer is that it is continuous. Consider for example the problem

$$x'' = H(t-3), \quad x(0) = x'(0) = 0.$$

Here, $f(t) = H(t-3)$ is discontinuous. The integral, or antiderivative, of a discontinuous function is continuous. Here, $x'(t) = \int H(t-3)dt = (t-3)H(t-3)$, which confirms this; integration is a smoothing process. Clearly, $x'(t)$ has a discontinuity in the derivative. Integrating again, we get $x(t) = \int (t-3)H(t-3)dt = \frac{1}{2}(t-2)^3 H(t-3)$. This is the solution of the equation and it is in fact differentiable. □

Example 3.21

In this example we calculate the response of an RC circuit when the emf is a discontinuous function. These types of problems occur frequently in engineering, especially electrical engineering, where discontinuous inputs to circuits are commonplace. Therefore, consider an RC circuit containing a 1 volt battery, and with zero initial charge on the capacitor. Take $R = 1$ and $C = \frac{1}{3}$. Assume the switch is turned on from $1 \le t \le 2$, and is otherwise switched off, giving a square pulse. Using $q = q(t)$ for the unknown charge on the capacitor, the governing is

$$q' + 3q = H(t-1) - H(t-2), \quad q(0) = 0.$$

We apply the basic technique with the notation $Q(s) = \mathcal{L}[q(t)]$. Taking the Laplace transform gives

$$sQ(s) - q(0) + 3Q(s) = \frac{1}{s}(e^{-s} - e^{-2s}).$$

Solving for $Q(s)$ yields

$$\begin{aligned} Q(s) &= \frac{1}{s(s+3)}(e^{-s} - e^{-2s}) \\ &= \frac{1}{s(s+3)}e^{-s} - \frac{1}{s(s+3)}e^{-2s}. \end{aligned}$$

Now we have to invert, which is always the hardest part. Each term on the right has the form $F(s)e^{-as}$, and therefore we can apply the switching property. From the table, or by partial fractions, we have

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+3)}\right] = \frac{1}{3}(1 - e^{-3t}).$$

Therefore, by the switching property,

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+3)}e^{-s}\right] = \frac{1}{3}(1 - e^{-3(t-1)})H(t-1).$$

Similarly,

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+3)}e^{-2s}\right] = \frac{1}{3}(1 - e^{-3(t-2)})H(t-2).$$

These two results give the charge on the capacitor as

$$q(t) = \frac{1}{3}(1 - e^{-3(t-1)})H(t-1) - \frac{1}{3}(1 - e^{-3(t-2)})H(t-2).$$

See Figure 3.4 □

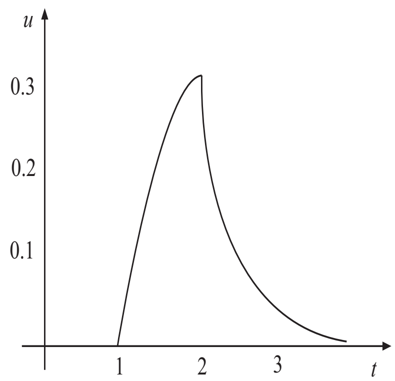

Figure 3.4 The switch is open up to time $t = 1$, so the charge response is zero. When the switch is closed at $t = 1$ the charge increases until $t = 2$, when the switch is again opened. The charge then decays to zero.

EXERCISES

1. Find a, b, and c for which

$$\frac{1}{s^2(s-1)} = \frac{as+b}{s^2} + \frac{c}{s-1}.$$

Then find the inverse Laplace transform of

$$\frac{1}{s^2(s-1)}.$$

2. Find the Laplace transform:

a) $H(t-\pi)\cos t$.

b) $3t^4$.

c) $t^2 \sin t$.

d) $\int_0^t \tau \cos\tau \, d\tau$.

e) $e^{-6t}\sin 3t$.

f) $e^{(a+bi)t}$.

g) $e^{2t} + (3 - e^2 t)H(t-1)$.

h) $(1-t)H(t-3)$.

3. Find the inverse transform of the following functions.

a) $\frac{s}{s^2+7s-8}$.

b) $\frac{3-2s}{s^2+2s+10}$.

c) $\frac{2}{(s-5)^4}$.

d) $\frac{1}{s(s-2)}e^{-s}$.

e) $\frac{7s+1}{s^2+4}$.

f) $\frac{3}{2s^2+7}$.

g) $\frac{4}{(s-3)^9}$.

h) $\frac{5s}{(s-3)^2+4}$.

i) $e^{-5s}\sin 2(t-3)$.

j) $e^{-s}\frac{s}{s^2+s+1}$

k) $\frac{1-e^{-2s}}{s^2}$.

l) $\frac{1}{s^4-1}$.

4. Write $f(t) = (t-1)H(t-1) - 2H(t-3) + e^{-t/2}H(t-4)$ as a multiline formula and plot $f(t)$ on $[0,\infty)$.

5. Plot the function

$$x(t) = \begin{cases} 6, & 0 \le t < 3; \\ 6e^{t-3}, & 3 < t < 4; \\ t-4, & 4 \le t < 6; \\ 0, & t \ge 6. \end{cases}$$

and find the Laplace tranform $X(s)$.

6. Solve the following initial value problems using Laplace transforms.

a) $x' + 5x = H(t-2), \quad x(0) = 1.$

b) $x' + x = \sin 2t, \quad x(0) = 0.$

c) $x'' - x' - 6x = 0, \quad x(0) = 2,\ x'(0) = -1$

d) $x'' - 2x' + 2x = 0, \quad x(0) = 0,\ x'(0) = 1.$

e) $x'' - 2x' + 2x = e^{-t}, \quad x(0) = 0,\ x'(0) = 1.$

f) $x'' - x' = 0, \quad x(0) = 1,\ x'(0) = 0.$

g) $x'' + 0.4x' + 2x = 1 - H(t-5), \quad x(0) = 0,\ x'(0) = 0.$

h) $x'' + 9x = \sin 3t, \quad x(0) = 0,\ x'(0) = 0.$

i) $x'' - 2x = 1, \quad x(0) = 1,\ x'(0) = 0.$

j) $x' = 2x + H(t-1), \quad x(0) = 0.$

7. Sketch the function $f(t) = 2H(t-3) - 2H(t-4)$ and find its Laplace transform.

8. Find the Laplace transform of $f(t) = t^2H(t-3)$.

9. Invert $F(s) = \frac{1}{(s-2)^4}$.

10. Find the inverse transform of

$$F(s) = \frac{1-e^{-4s}}{s^2}.$$

11. Solve the initial value problem

$$x'' + 4x = \begin{cases} \cos 2t, & 0 \le t \le 2\pi, \\ 0, & t > 2\pi, \end{cases}$$

where $x(0) = x'(0) = 0$. Sketch the solution.

12. Consider the initial value problem $x' = x + f(t)$, $x(0) = 1$, where $f(t)$ is given by

$$f(t) = \begin{cases} 0, & 0 < t \le 1 \\ -2, & t > 1. \end{cases}$$

Solve this problem in two ways: **(a)** by solving the problem on two intervals and pasting together the solutions in a continuous way, and **(b)** by Laplace transforms.

13. An LC circuit with $L = C = 1$ is *ramped-up* with an applied voltage

$$e(t) = \begin{cases} t, & 0 \le t \le 9 \\ 9, & t > 9. \end{cases}$$

Initially there is no charge on the capacitor and no current. Find and sketch a graph of the voltage response on the capacitor.

14. Solve $x' = -x + H(t-1) - H(t-2)$, $x(0) = 1$.

15. Solve the initial value problem

$$x'' + \pi^2 x = \begin{cases} \pi^2, & 0 < t < 1, \\ 0, & t > 1, \end{cases}$$

where $x(0) = 1$ and $x'(0) = 0$.

16. Let $f(t)$ be a periodic function with period p. That is, $f(t+p) = f(t)$ for all $t > 0$. Show that the Laplace transform of f is given by

$$F(s) = \frac{1}{1 - e^{-ps}} \int_0^p f(r) e^{-rs} dr.$$

Hint: Break up the interval $[0, +\infty)$ into subintervals $(np, (n+1)p)$, calculate the transform on each subinterval; finally use the geometric series $1 + z + z^2 + z^3 + \cdots = 1/(1-z)$.

17. Show that the Laplace transform of the periodic, square-wave function that takes the value 1 on intervals $[0, a)$, $[2a, 3a)$, $[4a, 5a)$,..., and the value -1 on the intervals $[a, 2a)$, $[3a, 4a)$, $[5a, 6a)$, ..., is

$$\frac{1}{s} \tanh\left(\frac{as}{2}\right).$$

18. Write a single-line formula for the function that is 2 between $2n$ and $2n+1$, and 1 between $2n-1$ and $2n$, where $n = 0, 1, 2, 3\ldots.$

19. Show that
$$\mathcal{L}\left[\int_0^t f(r)dr\right] = \frac{F(s)}{s}.$$
Hint: Take the transform of the time derivative of the integral.

20. Derive the formulas
$$\mathcal{L}\left[tf(t)\right] = -F'(s), \quad \mathcal{L}^{-1}[F'(s)] = -tf(t).$$
Hint: Calculate the derivative of $F(s)$.

21. Use the preceding exercise to find the inverse transform of $\arctan\left(\frac{a}{s}\right)$.

22. Show that
$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n F^{(n)}(s),\ n = 1, 2, 3, \ldots.$$

23. Show that
$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(r)dr,$$
and use the result to find
$$\mathcal{L}\left[\frac{\sinh t}{t}\right].$$

24. The Laplace transform applies to two (or more) simultaneous first-order differential equations with constant coefficients, for example,
$$x' = x - 2y$$
$$y' = 3x + y,$$
with initial conditions $x(0) = 1,\ y(0) = 0$. Here, $x = x(t)$ and $y = y(t)$ are the two unknowns. Letting $\mathcal{L}[x] = X(s)$ and $\mathcal{L}[y] = Y(s)$, solve the initial value problem for this pair of DEs.

25. Use Laplace transforms to solve the two simultaneous differential equations
$$x' = 2x - y$$
$$y' = x,$$
with $x(0) = a,\ y(0) = 0$.

3.3 The Convolution Property

The additivity property of Laplace transforms was stated earlier: the Laplace transform of a sum is the sum of the transforms. But what can we say about the Laplace transform of a product of two functions? Stated more precisely, if $x = x(t)$ and $y = y(t)$ with $\mathcal{L}[x] = X(s)$ and $\mathcal{L}[y] = Y(s)$, then what is $\mathcal{L}[xy]$? It is **not equal** to $X(s)Y(s)$. Then what is true? We ask it this way. What function has transform $X(s)Y(s)$, or differently, what is the inverse transform of $X(s)Y(s)$? The answer may be a surprise because it is nothing one would easily guess. The function whose transform is $X(s)Y(s)$ is the *convolution* of the two functions $x(t)$ and $y(t)$. This is defined as follows. If $x(t)$ and $y(t)$ are two functions defined on $[0, \infty)$, the **convolution** of x and y, denoted by $x * y$, is the function defined by

$$(x * y)(t) = \int_0^t x(\tau)y(t-\tau)d\tau.$$

Sometimes it is convenient to write the convolution as $x(t) * y(t)$. The **convolution property** of Laplace transforms states that

$$\mathcal{L}[x * y] = X(s)Y(s).$$

In terms of the inverse transform,

$$\mathcal{L}^{-1}[X(s)Y(s)] = (x * y)(t).$$

This property is very useful because when solving a DE we often end up with a product of transforms; we may use this last expression to invert the product.

Example 3.22

Find the convolution of 1 and t^2. We have

$$\begin{aligned} 1 * t^2 &= \int_0^t 1 \cdot (t-\tau)^2 d\tau = \int_0^t (t^2 - 2t\tau + \tau^2)d\tau \\ &= t^2 \cdot t - 2t(\frac{t^2}{2}) + \frac{t^3}{3} = \frac{t^3}{3}. \end{aligned}$$

Notice also that the convolution of t^2 and 1 is

$$t^2 * 1 = \int_0^t \tau^2 \cdot 1 d\tau = \frac{t^3}{3}. \quad \square$$

Remark 3.23

It can be shown (see the Exercises) in general that

$$(x * y)(t) = (y * x)(t),$$

and therefore the operation of convolution is commutative. This means that the integrand in the convolution can be arranged so that the choice of the shifted function makes the integration easier. In the last example, note that the second choice, where 1 is shifted is a simpler calculation. $\square$.

Verification of the convolution property is straightforward using the multivariable calculus technique of interchanging the order of integration. The reader should verify each of the following steps.

$$\begin{aligned}
\mathcal{L}\left(\int_0^t x(\tau)y(t-\tau)d\tau\right) &= \int_0^\infty \left(\int_0^t x(\tau)y(t-\tau)d\tau\right) e^{-st}dt \\
&= \int_0^\infty \left(\int_0^t x(\tau)y(t-\tau)e^{-st}d\tau\right) dt \\
&= \int_0^\infty \left(\int_\tau^\infty x(\tau)y(t-\tau)e^{-st}dt\right) d\tau \\
&= \int_0^\infty \left(\int_\tau^\infty y(t-\tau)e^{-st}dt\right) x(\tau)d\tau \\
&= \int_0^\infty \left(\int_0^\infty y(r)e^{-s(r+\tau)}dr\right) x(\tau)d\tau \\
&= \int_0^\infty \left(\int_0^\infty y(r)e^{-sr}dr\right) e^{-s\tau}x(\tau)d\tau \\
&= \left(\int_0^\infty e^{-s\tau}x(\tau)d\tau\right)\left(\int_0^\infty y(r)e^{-sr}dr\right). \square
\end{aligned}$$

This last expression is $X(s)Y(s)$. $\square$

In the exercises you are asked to show that $u * v = v * u$, so the order of the two functions under convolution does not matter.

Example 3.24

Find the inverse of

$$X(s) = \frac{3}{s(s^2+9)}.$$

We can do this by partial fractions, but here we use convolution. We have

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{3}{s(s^2+9)}\right] &= \mathcal{L}^{-1}\left[\frac{1}{s}\frac{3}{(s^2+9)}\right] \\ &= 1 * \sin 3t = \int_0^t \sin 3\tau d\tau \\ &= \frac{1}{3}(1-\cos 3t). \quad \square\end{aligned}$$

Example 3.25

Solve the nonhomogeneous DE

$$x'' + k^2 x = f(t),$$

where f is any given input function, and where $x(0)$ and $x'(0)$ are specified initial conditions. Taking the Laplace transform,

$$s^2 X(s) - sx(0) - x'(0) + k^2 X(s) = F(s).$$

Then

$$X(s) = x(0)\frac{s}{s^2+k^2} + x'(0)\frac{1}{s^2+k^2} + \frac{F(s)}{s^2+k^2}.$$

Now we can invert each term, using the table to calculate the inverse of the first two terms, and using convolution on the last term, to get the solution formula

$$x(t) = x(0)\cos kt + \frac{x'(0)}{k}\sin kt + \frac{1}{k}\int_0^t f(\tau)\sin k(t-\tau)dr.$$

Use of the convolution is a convenient way to find the solution to a differential equation with arbitrary source term. $\square$

EXERCISES

1. Compute the following convolutions:

   a) $\sin t * \cos t$.

   b) $e^{-2t} * e^{-3t}$.

   c) $t * t^2$.

   d) $t * e^t$.

2. Give a specific example to show that, in general, $\mathcal{L}[x(t)y(t)] \neq X(s)Y(s)$.

3. **(a)** Using Laplace transforms and the convolution property, find the general solution of the initial value problem

$$x' - ax = f(t), \quad x(0) = x_0.$$

using Laplace transforms. **(b)** Solve the equation using integrating factors, and compare with part (a).

4. Find the Laplace transform of the following integrals.

a) $f(t) = \int_0^t e^s(t-s)\,ds.$ b) $f(t) = \int_0^t e^{-(t-s)}\sin s\,ds.$

5. Use a change of variables in the convolution integral to show that the order of the functions used in convolution integral does not matter. That is,

$$(x * y)(t) = (y * x)(t).$$

6. Solve the initial value problem

$$x'' - \omega^2 x = f(t), \quad x(0) = x'(0) = 0.$$

7. Solve the initial value problem

$$x'' - 4x = 1 - H(t-1), \quad x(0) = x'(0) = 0.$$

8. Express the solution of the IVP as a convolution integral:

$$x'' + 3x' + 2x = e^{-4t}, \quad x(0) = x'(0) = 0.$$

9. Express the solution of the IVP as a convolution integral:

$$x'' + 2x' + 2x = \sin\omega t, \quad x(0) = x'(0) = 0.$$

10. Write an integral expression for the inverse transform of $X(s) = \frac{1}{s}e^{-3s}F(s)$, where $\mathcal{L}[f] = F$.

11. Find a formula for the solution to the initial value problem

$$x'' - x' = f(t), \quad x(0) = x'(0) = 0.$$

12. Use convolution to calculate

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right]$$

.

13. Use convolution to calculate

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right]$$

.

14. (**Integral Equations**) An integral equation is an equation where the unknown function $x(t)$ appears under an integral sign. In this exercise we consider integral equations having the form

$$x(t) = f(t) + \int_0^t k(t-\tau)x(\tau)d\tau,$$

where f and k, called the *kernel*, are given functions. These types of equations are called equations with a convolution kernel. Find a formula for the transform of the unknown, $X(s)$, in terms of the transforms F and K of f and k, respectively.

15. Using the method in the preceding exercise, solve the following integral equations.

a) $x(t) = t - \int_0^t (t-\tau)x(\tau)d\tau.$

b) $x(t) = 1 + \frac{1}{2}\int_0^t x(\tau)d\tau.$

c) $x(t) = \int_0^t x(\tau)d\tau + e^{-t}.$

d) $x(t) = -2\int_0^t \cos(t-r)x(r)dr.$

16. Solve the differential integral equation by Laplace transforms:

$$x'(t) - \frac{1}{2}\int_0^t (t-\tau)^2 x(\tau)\,d\tau = -t, \quad x(0) = 1.$$

17. Solve the integral equation for $x(t)$:

$$x(t) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{x(\tau)}{\sqrt{t-\tau}}\,d\tau.$$

Hint: Use the Gamma function introduced in Section 3.1.

3.4 Impulsive Sources

Many physical and biological processes have source terms that act at a single instant of time. For example, we can idealize an injection of medicine (a “shot”) into the blood stream as occurring at a single instant; a mechanical system, for example, a damped spring–mass system in a shock absorber on a car can be given an impulsive force by hitting a bump in the road; the switch in an electrical circuit can be closed only for an instant, which leads to an impulsive, applied voltage.

To fix the idea, let us consider a particle of mass m moving along a line for $t > 0$ and subject to a damping force equal to the velocity v and another applied force of magnitude $f(t)$; assume the particle has no initial velocity. By Newton's second law of motion,

$$mv' + v = f(t), \quad v(0) = 0.$$

This is a first-order linear equation, and if the force $f(t)$ is a continuous function, or piecewise continuous function, the problem can be solved by the methods presented in Chapter 2 (integrating factors) or by transform methods. We use the latter as illustration. Let $V(s) = \mathcal{L}[v(t)]$. Taking Laplace transforms of the equation gives

$$V(s) = \frac{1/m}{s + 1/m} F(s),$$

where $F(s)$ is the transform of the applied force $f(t)$. Using the convolution theorem,

$$v(t) = \frac{1}{m} \int_0^t e^{-(t-\tau)/m} f(\tau) d\tau. \tag{3.9}$$

We want to consider a special type of applied force $f(t)$, one given by an impulse that acts only for a single instant of time (i.e., think of the mass hit by a swift blow of a hammer). To fix the idea, we start the clock at $t = 0$ and the particle remains motionless until an impulse of 1 force unit is applied at the single instant of time $t = a$. We denote this unit impulsive force by $f(t) = \delta_a(t)$, which is called a **unit impulse** at $t = a$. The question is how to define it. Intuitively, it appears that we should take $\delta_a(t) = 1$ if $t = a$, and $\delta_a(t) = 0$, if $t \neq a$. But this cannot be correct. To illustrate, we substitute into (3.9) and obtain

$$v(t) = \frac{1}{m} \int_0^t e^{-(t-\tau)/m} \delta_a(\tau)\, d\tau. \tag{3.10}$$

If $\delta_a(t) = 0$ at all values of t, except $t = a$, the integral must be zero because the integrand is zero except at a single point. Therefore, the velocity is $v(t) = 0$ for all t, which is incorrect. Something is wrong with this intuitive definition of the unit impulse $\delta_a(t)$.

The problem is we have not yet to come to terms with the idea of an impulse, so let us take a different approach. Let $p(t) = mv(t)$ be the momentum of the mass. In general, Newton's law states that the time rate of change of momentum is the force, or

$$\frac{dp}{dt} = f(t).$$

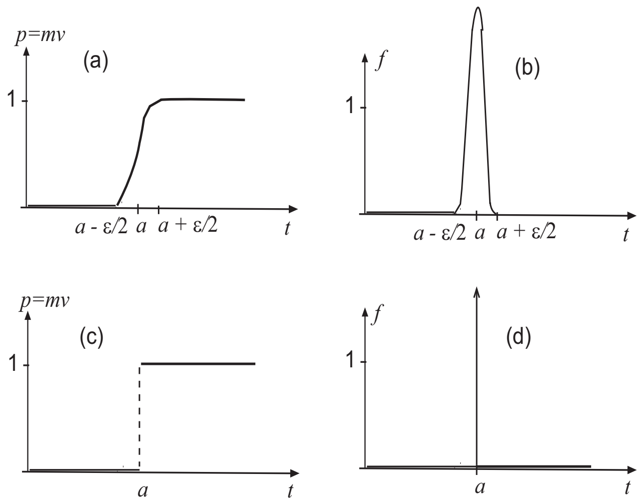

Figure 3.5 (a) The momentum changes from 0 to 1 over a small time interval $(a-\varepsilon/2, a+\varepsilon/2)$, and (b) the force causing this change; (c) the idealized case when the change in momentum occurs at a single instant of time $t=a$, and (d) the resulting impulsive force causing this change.

In elementary physics, an impulse is defined as the change of momentum Δp that occurs when a force acts over a small instant of time Δt. Thus, the impulse is $\Delta p = f(t)\Delta t$. If the unit impulse is centered at $t=a$, and the force acts over a small time interval $(a-\varepsilon/2, a+\varepsilon/2)$, then we can imagine that the momentum changes from 0 to 1. This situation is depicted in the plot shown in Figure 3.5, panel (a). The resulting force, which is the derivative of momentum, has the shape shown in panel (b). However, we always have

$$\Delta p = \int_{a-\varepsilon/2}^{a+\varepsilon/2} f(t)dt = 1.$$

Notice that as ε gets smaller and smaller, the last relation still holds true. Mathematically, we idealize this situation and assume the last relation holds true in the limit as $\varepsilon \to 0$. Thus, the momentum changes abruptly, as shown in Figure 3.5, panel (c). The corresponding derivative, or idealized applied force, is shown in 3.5, panel (d). The force just acts as a point source at $t=a$. But the change in momentum is still 1.

In summary, we regard a point source as a mathematical idealization. Let us take this idealization further and consider an applied unit force acting at

$t = a$ and given by

$$f_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & a - \frac{\varepsilon}{2} < t < a + \frac{\varepsilon}{2} \\ 0, & \text{otherwise,} \end{cases}$$
$$= \frac{1}{\varepsilon}\left[H\left(t - \left(a - \frac{\varepsilon}{2}\right)\right) - H\left(t - \left(a + \frac{\varepsilon}{2}\right)\right)\right].$$

where H is the Heaviside function. These idealized forces are rectangular inputs

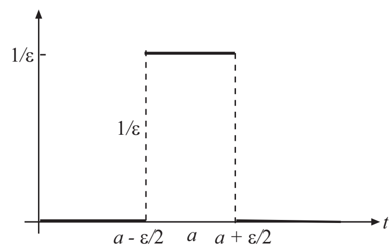

Figure 3.6 The idealized (rectangular) impulsive force $f_{a,\varepsilon}(t)$ of height $1/\varepsilon$ and width ε. As $\varepsilon \to 0$, the function gets narrower and higher, but always has area 1.

that get taller and narrower (of height $1/\varepsilon$ and width ε) as ε gets small, and for any ε we

$$\int_{a-\varepsilon/2}^{a+\varepsilon/2} f_\varepsilon(t)dt = 1.$$

It seems reasonable therefore to define the unit impulse $\delta_a(t)$ at $t = a$ in a limiting sense, having the property

$$\int_{a-\varepsilon/2}^{a+\varepsilon/2} \delta_a(t)dt = 1, \quad \text{for all } \varepsilon > 0.$$

Engineers and scientists used this condition, along with $\delta_a(t) = 0, \;\; t \neq a$, for decades to define a unit, point source at time $t = a$, called the **delta function**, and they developed a calculus that was successful in obtaining solutions to equations having point sources. But, actually, the unit impulsive force is not a function at all, and it was shown in the mid-twentieth century that the unit impulse belongs to a class of so-called *generalized functions* whose actions are not defined pointwise, but rather by how they act when integrated against other functions. Mathematically, the unit impulse $\delta_a(t)$ is defined by the **sifting property**

$$\int_0^\infty \delta_a(t)\phi(t)dt = \phi(a).$$

That is, when integrated against any nice function $\phi(t)$, the delta function picks out the value of $\phi(t)$ at $t = a$. On the other hand, over a variable interval we have

$$\int_0^t \delta_a(t)\phi(t)dt = \begin{cases} 0, & t < a; \\ \phi(a), & t > a. \end{cases}$$

or,

$$\int_0^t \delta_a(\tau)\phi(t-\tau)dt = H(t-a)\phi(a).$$

We check that this works in our problem. If we use this sifting property back in (3.10), then for $t > a$ the velocity is

$$v(t) = \frac{1}{m}\int_0^t e^{-(t-\tau)/m}\delta_a(\tau)d\tau = \frac{1}{m}H(t-a)e^{-(t-a)/m},$$

which is the correct solution. That is, $v(t) = 0$ up until the time $t = a$, because there is no force. Furthermore, $v(a) = 1/m$. Therefore the velocity is zero up to time $t = a$, at which it jumps to the value $1/m$, and then decays away.

To deal with differential equations involving impulses we can use Laplace transforms in a formal way. Using the sifting property, with $\phi(t) = e^{-st}$, we obtain

$$\mathcal{L}[\delta_a(t)] = \int_0^\infty \delta_a(t)e^{-st}dt = e^{-as},$$

which is a formula for the Laplace transform of the unit impulse function. This gives, of course, the inverse formula

$$\mathcal{L}^{-1}[e^{-as}] = \delta_a(t).$$

If the impulse is given at $t = a = 0$, then

$$\mathcal{L}[\delta_0(t)] = \int_0^\infty \delta_0(t)e^{-st}dt = 1.$$

This gives the inverse formula

$$\mathcal{L}^{-1}[1] = \delta_0(t).$$

The previous discussion is highly intuitive and lacks a careful mathematical base. However, the ideas can be made precise and rigorous. We refer to advanced texts for a thorough treatment of generalized functions. Another common notation for the unit impulse $\delta_a(t)$ is $\delta(t-a)$. If an impulse has magnitude f_0, instead of 1, then we denote it by $f_0\delta_a(t)$. For example, an impulse given to a mass of magnitude 12 at time $t = a$ is $12\delta_a(t)$.

Next, we present another calculation of the Laplace transform of the unit impulse function. The idea is to compute the transform of the idealized impulse $f_\varepsilon(t)$ (see Figure 3.6), and then take the limit as $\varepsilon \to 0$. We have

$$\begin{aligned}
\mathcal{L}[f_\varepsilon(t)] &= \mathcal{L}\left[\frac{1}{\varepsilon}\left[H\left(t-\left(a-\frac{\varepsilon}{2}\right)\right) - H\left(t-\left(a+\frac{\varepsilon}{2}\right)\right)\right]\right] \\
&= \frac{1}{\varepsilon s}e^{-(a-\varepsilon/2)s} - \frac{1}{\varepsilon s}e^{-(a+\varepsilon/2)s} \\
&= \frac{1}{\varepsilon s}e^{-as}\left(e^{\varepsilon s/2} - e^{-\varepsilon s/2}\right) \\
&= \frac{1}{\varepsilon s}e^{-as}\frac{2\sinh\frac{\varepsilon s}{2}}{\varepsilon} \\
&= e^{-as}\frac{\sinh(\varepsilon s/2)}{\varepsilon s/2}.
\end{aligned}$$

Using l'Hospital's rule to compute the limit,

$$\lim_{\varepsilon\to 0}\frac{\sinh(\varepsilon s/2)}{\varepsilon s/2} = \lim_{\varepsilon\to 0}\cosh(\varepsilon s/2) = 1.$$

Therefore we have

$$\lim_{\varepsilon\to 0}\mathcal{L}[f_\varepsilon(t)] = e^{-as},$$

or

$$\mathcal{L}[\delta_a(t)] = e^{-as}.$$

Example 3.26

Solve the initial value problem

$$x'' + x' = \delta_2(t), \quad x(0) = x'(0) = 0,$$

with a unit impulse applied at time $t = 2$. Taking the transform,

$$s^2X(s) + sX(s) = e^{-2s}.$$

Thus

$$X(s) = \frac{e^{-2s}}{s(s+1)}.$$

Using the table it is simple to find

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] = 1 - e^{-t}.$$

Therefore, by the shift property, the solution is

$$x(t) = \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s(s+1)}\right] = (1 - e^{-(t-2)})H(t-2).$$

Alternately, we can obtain $x(t)$ using the convolution property and delta functions. We have

$$X(s) = \frac{1}{s(s+1)}e^{-2s},$$

and by convolution,

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] * \mathcal{L}^{-1}[e^{-2s}] \\ &= (1-e^{-t}) * \delta_2(t) \\ &= \int_0^t (1-e^{-(t-\tau)})\delta_2(\tau)\,d\tau \\ &= H(t-2)\left(1-e^{-(t-2)}\right). \end{aligned}$$

The initial conditions are zero, and so the solution is zero up until time $t = 2$, when the impulse occurs. At that time the solution increases with limit 1 as $t \to \infty$. See Figure 3.7. □

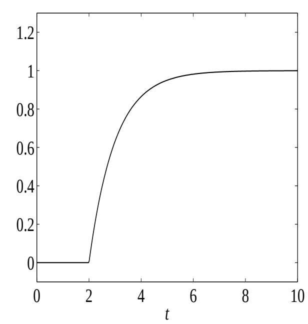

Figure 3.7 Solution in Example 3.26: $x(t) = H(t-2)\left(1-e^{-(t-2)}\right)$.

An Input–Output Approach

Many problems occurring in electrical and chemical engineering, as well as in other areas, can be characterized as *input-output systems*. See Figure 3.8. These systems can be quite complicated with many components with many inputs and many outputs. For example, in an industrial chemical process the system could be a sequence of reactors, even with feedbacks among the components to control the various processes.

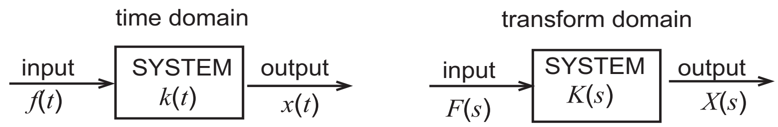

Figure 3.8 Input-output systems in the time domain and the transform domain.

A linear nonhomogeneous differential equation

$$ax'' + bx' + cx = f(t), \tag{3.11}$$

$$x(0) = x'(0) = 0. \tag{3.12}$$

is the simplest example of an input-output system; $f(t)$ is the input and the solution $x(t)$, which is called the forced response, is the output. We have taken zero initial conditions, although it is not necessary. Of particular interest is understanding how such systems respond to special, representative inputs, or loads, such as an impulse, a step function, and a ramped load, shown in Figure 3.9. Such systems can be studied in either the time domain, or the transform domain.

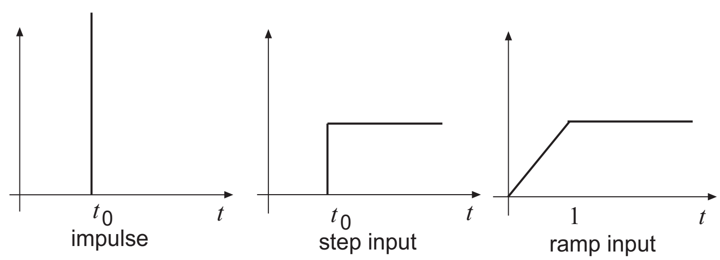

Figure 3.9 Sample inputs: an impulse, $\delta_{t_0}(t)$; a step function, $H(t - t_0)$; ramped loading, $t + (1 - t)H(t - 1)$.

Let's consider the input-output system defined in (3.11)–(3.12) and examine it in the transform domain. From our earlier treatment the forced response is

$$X(s) = K(s)F(s),$$

where $F(s) = \mathcal{L}[f(t)]$ and

$$K(s) = \frac{1}{as^2 + bs + c}, \qquad \text{(transfer function)}$$

which is called the **transfer function**. Knowing the transfer function gives complete knowledge of the response of the system. It follows by the convolution property that

$$x(t) = k(t) * f(t) = \int_0^t k(t-\tau)f(\tau)\,d\tau,$$

which is the forced response. Note the similar, but contrasting representations of the forced response in the time domain and the transform domain:

$$x(t) = k(t) * f(t), \qquad X(s) = K(s)F(s).$$

Example 3.27

(**Impulse response**) If $f(t) = \delta_{t_0}(t)$, then in the transform domain the impulse response

$$X(s) = K(s)e^{-t_0 s}.$$

In the time domain,

$$\begin{aligned} x(t) &= k(t) * \delta_{t_0}(t) \\ &= \int_0^t k(t-\tau)\delta_{t_0}(\tau)\,d\tau \\ &= H(t-t_0)k(t-t_0). \quad \square \end{aligned}$$

Example 3.28

Find the impulse response:

$$x'' + 2x' + 5x = \delta_{t_0}(t), \quad x(0) = x'(0) = 0.$$

The transfer function is

$$K(s) = \frac{1}{s^2+2s+5} = \frac{1}{2}\frac{2}{(s+1)^2+4}$$

and therefore

$$X(t) = \frac{1}{2}\frac{2}{(s+1)^2+4}\,e^{-t_0 s}.$$

Taking inverse transforms and using convolution,

$$\begin{aligned} x(t) &= \ e^{-t}\sin 2t * \delta_{t_0}(t) \\ &= \frac{1}{2}\int_0^t e^{-(t-\tau)}\sin 2(t-\tau)\delta_{t_0}(\tau)\,d\tau \\ &= \frac{1}{2}H(t-t_0)e^{-(t-t_0)}\sin 2(t-t_0). \end{aligned}$$

This impulse response is zero up to time $t = t_0$ and then is a damped oscillation for $t > t_0$. See Figure 3.10 for $t_0 = 2$. $\square$

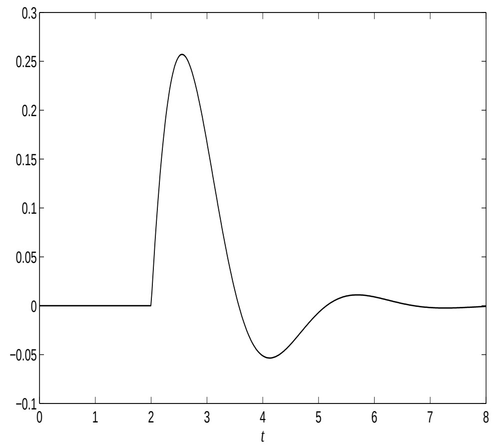

Figure 3.10 Plot of the impulse response $x(t) = \frac{1}{2}H(t-2)e^{-t(t-2)}\sin 2(t-2)$.

EXERCISES

1. Compute $\int_0^\infty e^{-2(t-3)^2}\delta_4(t)dt$.
2. Solve the initial value problem
$$x' + 3x = \delta_1(t) + H(t-4),$$
$$x(0) = 1.$$
Sketch the solution.
3. Solve the initial value problem
$$x'' - x = \delta_5(t),$$
$$x(0) = x'(0) = 0.$$
Sketch the solution.
4. Solve the initial value problem
$$x'' + x = \delta_2(t),$$
$$x(0) = x'(0) = 0.$$
Sketch the solution.
5. Invert the transform $F(s) = e^{-2s}/s + e^{-3s}$.

6. Solve the initial value problem

$$x'' + 4x = \delta_2(t) - \delta_5(t),$$
$$x(0) = x'(0) = 0.$$

7. Solve the initial value problem

$$x'' + x = 3\delta_{2\pi}(t),$$
$$x(0) = 0,\ x'(0) = 1.$$

8. Consider a spring–mass setup with $m = k = 1$, where k is the spring constant. Initially the system is at rest, at equilibrium. At each of the times $t = 0,\ \pi,\ 2\pi,\ 3\pi, ..., n\pi, ...$ a unit impulse is given to the mass. Determine the resulting displacement $x(t)$ of the mass.

9. Solve the problem

$$2x'' + x' + 2x = \delta_5(t), \quad x(0) = x'(0) = 0.$$

(a) Identify the transfer function. **(b)** Find the impulse response in both the time and transform domain. **(c)** Plot the impulse response for $0 \le t \le 20$.

10. Solve

$$y'' + y' + y = \delta_1(t), \quad y(0) = y'(0) = 0,$$

and plot the solution.

11. A purely oscillatory system is ramped loaded at $t = 5$ with the input

$$f(t) = \frac{1}{5}(t-5)H(t-5) + \left(1 - \frac{1}{5}(t-5)\right)H(t-10).$$

Find the response if the system is

$$x'' + 4x = f(t), \quad x(0) = x'(0) = 0,$$

and plot the graph of $x(t)$.

Table 3.1 Short Table of Laplace Transforms

$x(t)$	$X(s)$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}, \quad n = 0, 1, 2, 3, ...$
$f(t)e^{at}$	$F(s-a)$
$H(t-a)$	$\frac{1}{s}e^{-as}$
$H(t-a)(t)f(t-a)$	$e^{-as}F(s)$
$f(t)H(t-a)$	$e^{-as}\mathcal{L}[f(t+a)]$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$t\sin kt$	$\frac{2bs}{(s^2+k^2)^2}$
$t\cos kt$	$\frac{s^2-b^2}{(s^2+k^2)^2}$
$\frac{1}{a-b}(e^{at}-e^{bt})$	$\frac{1}{(s-a)(s-b)}$
$x^{(n)}(t)$	$s^nX(s) - s^{n-1}x(0) - \cdots - x^{(n-1)}(0)$
$f(at)$	$\frac{1}{a}F(\frac{s}{a})$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$
$\delta_a(t)$	e^{-as}
$\delta_0(t)$	1
$\int_0^t f(\tau)g(t-\tau)d\tau$	$F(s)G(s)$
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
$tf(t)$	$-F'(s)$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$
$\sum_0^\infty f(t-na)H(t-na)$	$F(s)\frac{1}{1-e^{as}}$

4
Linear Systems

The first three chapters dealt a single first-order or second-order linear differential equation. A natural next step is to examine coupled systems of differential equations with several unknown functions. Many problems from the pure and applied sciences have multiple components linked together in some manner, and it is natural to formulate those as first-order systems. For example, two chemical reactors may be coupled together with two concentrations to account for, one in each reactor. Or an electrical circuit may have coupled loops, each carrying a different current. Predator-prey models in ecology involve two animal species, the prey and predator, and their populations are coupled together through their interactions.

In this text we consider only planar, or two-dimensional systems, consisting of two first-order equations in two unknown functions $x(t)$ and $y(t)$. As we will see, second-order equations, studied in Chapter 2, are equivalent to such a planar system, but a systems approach exposes an insightful geometrical structure revealing the dynamics. Our goal is to show what to expect from a system, including general solutions and how to display them graphically for easy visualization. This is carried out by introducing matrix notation, a language that greatly simplifies their representation and which extends in a straightforward way to systems of higher dimensions.

4.1 Linear Systems vs. Second-Order Equations

Consider the damped oscillator equation

$$mx'' + \gamma x' + kx = 0$$

with unknown displacement $x = x(t)$. If we introduce the velocity $y = y(t)$ as another dependent function, then $x' = y$ and the damped oscillator equation becomes

$$my' + \gamma y + kx = 0,$$

or, $my' = -kx - \gamma y$. Therefore,

$$\begin{cases} x' = y, \\ y' = -\frac{k}{m}x - \frac{\gamma}{m}y, \end{cases}$$

This is a system of two first-order equations in two unknowns $x = x(t)$ and $y = y(t)$ and it is completely equivalent to the second-order damped oscillator equation.

Remark 4.1

We can always reduce a second-order linear equation

$$ax'' + bx' + cx = 0$$

to a first-order linear system by defining $y = x'$, as above. □

As well, first-order systems of equations occur naturally, as the next example shows.

Example 4.2

(**Compartmental Model**) Many linear systems come from compartmental models. These are models where there are several compartments with specified flow rates between them. For example, coupled chemical reactors, disease models (those susceptible and those infected), physiological models (a chemical in the blood and in an organ), and so forth, are all compartmental models. In this example we consider a farm crop and the surrounding soil as two compartments. If a pesticide is sprayed on the soil at time $t = 0$, then the pesticide transfers to the plants through uptake in the root systems. Conversely, the plants transfer the pesticide back to the soil through respiration processes. Added to this constant exchange of chemicals between the two compartments, there is simultaneously natural degradation of the pesticide in the soil. We visualize this

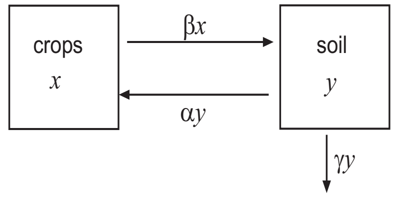

Figure 4.1 A compartmental diagram showing the exchange rates of the herbicide between crops and the soil. The $-\gamma y$ term represents degradation, or decay, in the soil.

processes by drawing a compartmental diagram as shown in Figure 4.1 indicating the flow rates between compartments. Let $x = x(t)$ be the amount of pesticide (mass per volume) in the crop, and $y = y(t)$ the amount in the soil. Further, let αy be the rate that the pesticide is taken up by the plants, and βx the rate that it is transferred back to the soil; assume the pesticide in the soil degrades naturally with γy the degradation rate. Attending to the signs of the terms, have

$$\begin{aligned}\frac{dx}{dt} &= \text{Rate lost by the crop} + \text{Rate gained by the crop}\\ &= -\beta x + \alpha y,\end{aligned}$$

and

$$\begin{aligned}\frac{dy}{dt} &= \text{Rate gained the soil} + \text{Rate lost by the soil}\\ &\quad + \text{Rate of degradation} = \beta x - \alpha y - \gamma y.\end{aligned}$$

Therefore, we obtain a system of two equations with unknowns $x(t)$ and $y(t)$:

$$\begin{aligned}\frac{dx}{dt} &= -\beta x + \alpha y,\\ \frac{dy}{dt} &= \beta x - (\alpha + \gamma) y.\end{aligned}$$

We examine this model later in detail. □

Notation and Terminology

A linear homogeneous system of differential equations with constant coefficients has the form

$$\frac{dx}{dt} = ax + by, \tag{4.1}$$

$$\frac{dy}{dt} = cx + dy, \tag{4.2}$$

where a, b, c, and d are constants, and where $x = x(t)$ and $y = y(t)$ are the unknown states. For conciseness we often use x' and y' for derivatives. A **solution** consists of a *pair* of functions

$$x = x(t), \quad y = y(t),$$

that, when substituted into the equations, reduces the equations to identities for all t. For planar systems of the form (4.1)–(4.2), solutions exist for all time $-\infty < t < \infty$. In applications we are generally interested in nonnegative times $t \geq 0$. There are *infinitely many solutions* to a linear system, and a single, unique solution is selected out if there are prescribed **initial conditions**, which take the form

$$x(0) = x_0, \quad y(0) = y_0, \qquad (x_0,\ y_0 \text{ given}).$$

We visualize a solution in two ways. We can plot both $x = x(t)$ and $y = y(t)$ versus t on the same set of axes, as in Chapters 1 and 2; these are the time series, or **component plots**, and they tell us how the states x and y vary in time. Or, we think of $x = x(t)$, $y = y(t)$ as **parametric equations** of a curve in an xy plane, with time t as the parameter along the curve; each value of t corresponds to a point $(x(t), y(t))$ in the plane which is traced out as t varies. Their direction as time increases is indicated by placing arrows on the curves. In the parametric context, a solution curve is called an **orbit**, and the xy plane is called the **phase plane**. Other words used to describe a solution curve or orbit are **path**, and **trajectory**. Figure 4.2 is an illustration of an orbit.

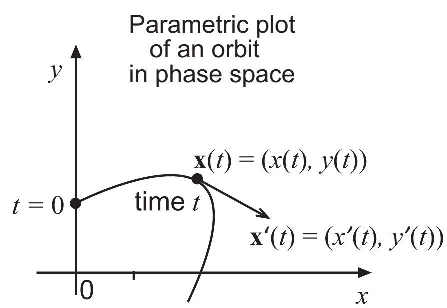

Figure 4.2 An orbit in the xy-phase plane represented by a moving point or the position vector $\mathbf{x}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} = (x(t), y(t))$. Equivalently we represent the vector solution as a column vector. The vector $\mathbf{x}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} = (x'(t), y'(t))$ is the velocity vector and is tangent to the curve at the point $(x(t), y(t))$ at time t, and it indicates the direction the curve is traced out in time.

We recall from multivariable calculus that the orbit is the path traced out in time by a particle defined by the *position vector* $\mathbf{x}(t) = x(t)\mathbf{i}+y(t)\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors. The vector $\mathbf{x}'(t) = x'(t)\mathbf{i}+y'(t)\mathbf{j}$ is the *velocity vector* and it represents the tangent vector to the orbit at time t, and its magnitude is the particle's speed. In differential equations we use a *column vector* representation of an orbit,

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

$x(t)$ is called the x-component and $y(t)$ is called the y-component of $\mathbf{x}(t)$. Its derivative vector, which gives components of the tangent vector, is then

$$\mathbf{x}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}.$$

It is also common to write a vector as an ordered pair $\mathbf{x}(t) = (x(t), y(t))$, which is an efficient notation for typed material. The velocity vector in this form is $\mathbf{x}'(t) = (x'(t), y'(t))$. This ordered-pair notation, based on the equivalence between points and vectors, should cause no confusion even though points and vectors are technically different objects.

We can take a geometrical approach to visualize the orbits of a system, similar to that in Chapter 1 where we visualized the solution curves of a single differential equation by plotting the slope field. For systems, at any point (x, y) in the xy phase plane, the right sides of (4.1)–(4.2) define the components of the tangent vector $\mathbf{F}$,

$$\mathbf{F}(x, y) = (x', y') = (ax + b, cx + d),$$

at that point. The orbit that goes through the point (x, y) must have tangent vector components (x', y'). We can plot, or have software plot for us, this vector at several points (x, y) in the phase plane to obtain an overall pattern, or structure, of the orbits of the system. The set of all the tangent vectors $\mathbf{F}(x, y)$ is called the **vector field** (a 'field' of vectors), indicating the direction of the orbits. The orbits fit in so that their tangents coincide with the vector field.

There are two special straight lines along which the tangent vector field is either vertical or horizontal; these are called the **nullclines**. The x-**nullcline** is the straight line $ax + by = 0$ where $x' = 0$, or the vector field is vertical. The y-**nullcline** is the straight line $cx + dy = 0$ where $y' = 0$, or the vector field is horizontal. Finding and plotting the two nullclines are essential tools for sketching orbits. A picture showing several key orbits is called the **phase diagram** of the system (4.1)–(4.2).

Example 4.3

For the system

$$x' = y,$$
$$y' = -4x,$$

the vector field at the point (x, y) is given by

$$\mathbf{F} = (y, -4x).$$

Figure 4.3 shows the vector field, computed using software, at a set of points in the xy plane. Easily we can visualize the the rotational nature of the orbits. It is shown in the next example that the orbits are a family of ellipses $x^2 + \frac{1}{4}y^2 = C$

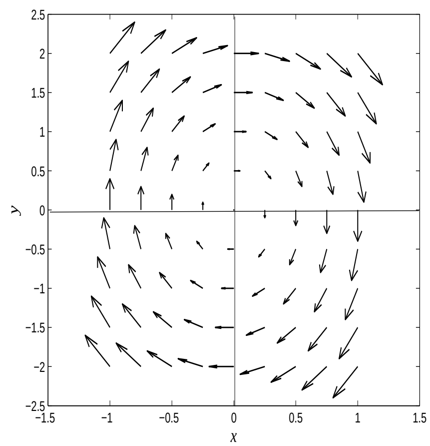

Figure 4.3 The vector field indicating a clockwise, elliptical flow of the system $x' = y, \;\; y' = -4x$ around the origin. The x-nullcline is the x axis ($y = 0$) where the vector field is vertical, and the y-nullcline is the y axis ($x = 0$) where the vector field is horizontal.

that fit into this field. Without software we can make the following argument. If $y > 0$ (upper half plane), then $x' > 0$ and therefore x must be increasing; if $y < 0$ (lower half plane) then $x' < 0$ and x must be decreasing. If $x > 0$ (right half plane), then $y' < 0$, so y must be decreasing; if $x > 0$ (left half plane), then $y' > 0$, so y must be increasing. Observe also that the vector field is vertical (up or down) along $x' = y = 0$, which is along the x axis; the x axis is the x-nullcline. The vector field is horizontal (right or left) along

$y' = -4x = 0$, which is the y axis; the y axis is the x-nullcline. This is all consistent with the direction field in Figure 4.3. Note, however, even though a vector field for a system has a rotational nature, its orbits may not be closed paths, like ellipses. Orbits that spiral around and into the origin could equally fit the field. Therefore the vector field itself does not tell the whole story, but it can supplement analytical calculations to get fine structure properties of orbits. □

In addition to the parametric representation of an orbit (i.e., the solution), there is another representation of an orbit as a relation between x and y, with time suppressed. If $x = x(t),\ y = y(t)$ are known, it may be found by eliminating the time parameter t in the parametric equations. For example, the set of parametric equations $x(t) = t$, $y(t) = 2t^2$, $-\infty < t < \infty$, becomes, when time is eliminated, $y = 2x^2$, which is a parabola. In this latter representation, we know the orbits shape, but we lose the information about how it is traced out in time.

Example 4.4

Consider the linear system

$$x' = y, \tag{4.3}$$

$$y' = -4x. \tag{4.4}$$

It is straightforward to check by simple substitution that

$$x(t) = \cos 2t, \quad y = -2\sin 2t,$$

or, equivalently,

$$\mathbf{x}(t) = \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix}$$

is a solution. Using software we can plot the components immediately, as shown in Figure 4.4 (left panel). Clearly, the solution is oscillatory with period π. We can also identify the orbit by eliminating the parameter t to obtain a relationship between the coordinates x and y. Note that $x^2 = \cos^2 2t$ and $y^2 = 4\sin^2 2t$; or $x^2 = \cos^2 2t,\ \ \frac{1}{4}y^2 = \sin^2 2t$. Adding the two equations gives

$$x^2 + \frac{1}{4}y^2 = 1,$$

which we recognize as an *ellipse* intersecting the coordinate axes at $(0, \pm 2)$ and $(1, \pm 0)$. This form of the solution tells does not indicate the time dependence. On the other hand, the graph of the solution form $\mathbf{x}(t)$ gives specific information how the orbit is swept out as a function of time. Initially, at $t = 0$ it is at $(1, 0)$

and winds clockwise to: $(0,-2)$ at $t=\pi/4$; $(-1,0)$ at $t=\pi/2$; $(0,2)$ at $t=3\pi/4$; and returning to $(1,0)$ at $t=\pi$. Therefore, one oscillation is completed in π units of time. We can always find the direction along the orbit by examining the direction field of the system, as in the last example. This shows the elliptical orbit is clockwise. □

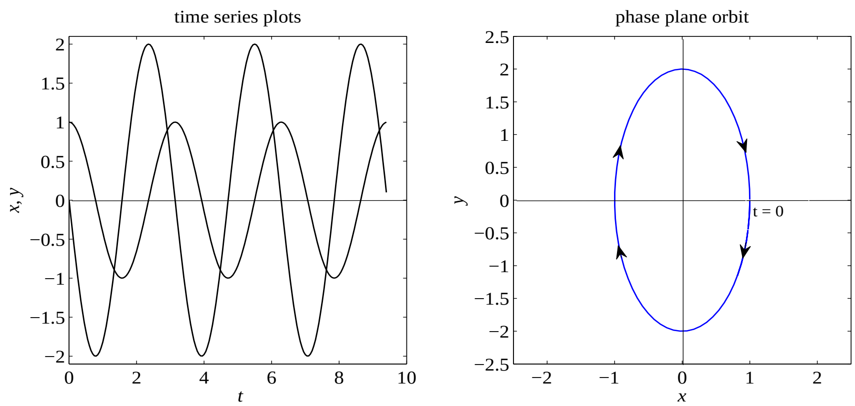

Figure 4.4 Two plots showing the two representations of a solution to a system (4.3)–(4.4) for $t \geq 0$. (Left) The component plots $x=\cos 2t$, $y=-2\sin 2t$, which are periodic, or oscillatory, of period π. (Right) The corresponding clockwise elliptical orbit in the xy-phase plane; it begins at $(1,0)$ at time $t=0$ and winds clockwise, making one revolution every π units of time.

Remark 4.5

(**The Zero Solution**) The linear system (4.1)–(4.2) always has the solution $x(t)=0$, $y(t)=0$, or $\mathbf{x}(t)=\mathbf{0}$, which is the *zero solution*. In the phase plane this orbit consists of a *single point*, $\mathbf{0}=(0,0)$, and the orbit remains there for all time. This solution is an **equilibrium solution** because it does not change in time. The point $(0,0)$ represented by this solution is called a **critical point**. Systems of differential equations can have other equilibria, or constant, solutions at other points; this important concept is discussed in Section 4.3. □

Method of Elimination

At the beginning of this section we showed that a second-order equation could be easily transformed to a system of two first-order equations. Now we show the converse—*a linear system can always be converted to a single second-order equation*.

Consider the linear system

$$x' = ax + by, \tag{4.5}$$

$$y' = cx + dy, \tag{4.6}$$

We can eliminate one of the variables, say, y, to obtain a second-order, linear differential equation with constant coefficients for $x = x(t)$. This takes 2 steps. (i) Take the derivative of the first equation, and then substitute y' from the second. (ii) Substitute for y from the first equation. Analytically,

$$\begin{aligned} x'' &= ax' + by' \\ &= ax' + b(cx + dy) \\ &= ax' + bcx + d(x' - ax). \end{aligned}$$

Rearranging terms, we obtain an equivalent second-order equation

$$x'' - (a + d)x' + (ad - cb)x = 0. \tag{4.7}$$

It is clear that we can solve the system (4.5)–(4.6) for $x = x(t)$, $y = y(t)$ by solving (4.7) for $x(t)$ and using (4.5) to find $y(t)$:

$$y(t) = \frac{1}{b}(x' - ax). \tag{4.8}$$

The method for solving (4.5)–(4.6) using (4.7)–(4.8) is called **elimination**. It gives us the form of the *general* solution to (4.5)–(4.6), and it indicates what to expect from any planar, linear system.

Example 4.6

(**Elimination**) Consider the simple linear system from the preceding example,

$$x' = y, \tag{4.9}$$

$$y' = -4x. \tag{4.10}$$

Taking the derivative of the (4.9) and then using (4.10) immediately gives

$$x'' = y' = -4x,$$

or

$$x'' + 4x = 0.$$

From Chapter 2, the general solution is oscillatory and has the form

$$x(t) = c_1 \cos 2t + c_2 \sin 2t.$$

Therefore, using $y = x'$, we get

$$y(t) = -2c_1 \sin 2t + 2c_2 \cos 2t.$$

Therefore the general solution to the system (4.3)–(4.4) in vector format is

$$\mathbf{x}(t) = \begin{pmatrix} c_1 \cos 2t + c_2 \sin 2t \\ -2c_1 \sin 2t + 2c_2 \cos 2t \end{pmatrix} = c_1 \begin{pmatrix} \cos 2t \\ -2 \sin 2t \end{pmatrix} + c_2 \begin{pmatrix} \sin 2t \\ 2 \cos 2t \end{pmatrix}. \ \square$$

Remark 4.7

Note the form of the solution in Example 4.6. In vector format it is the linear combination of two vector solutions

$$\mathbf{x}_1(t) = \begin{pmatrix} \cos 2t \\ -2 \sin 2t \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} \sin 2t \\ 2 \cos 2t \end{pmatrix}.$$

This example shows what to expect from any linear system: the general solution is a linear combination of two basic vector solutions. □

Next we state some fundamental theorems similar to those in Chapter 2 for second-order equations. To proceed we need some terminology. If two solutions

$$\mathbf{x}_1(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}$$

have the property that their components satisfy

$$W(t) \equiv \phi_1(t)\psi_2(t) - \phi_2(t)\psi_1(t) \neq 0 \quad \text{for all} \quad t,$$

then we say $\mathbf{x}_1(t),\ \mathbf{x}_2(t)$ is a **fundamental set** of solutions. The function $W(t)$ is called the **Wronskian** of $\mathbf{x}_1(t),\ \mathbf{x}_2(t)$. The non-vanishing of the Wronskian for all t is a way of saying the two solutions are independent. For example, taking the two solutions Remark 4.7, the Wronskian is

$$W(t) = (\cos 2t)(2\cos 2t) - (-2\sin 2t)(\sin 2t) = 2\cos^2 2t + 2\sin^2 2t = 2 \neq 0$$

for all t. Therefore these two solutions are linearly independent and form a fundamental set.

Theorem 4.8

(**Superposition**) If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are any two vector solutions of the linear system (4.5)–(4.6), then $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ is a solution for any constants c_1 and c_2. □

Theorem 4.9

(**Uniqueness**) The initial value problem

$$x' = ax + by, \tag{4.11}$$

$$y' = cx + dy, \tag{4.12}$$

$$x(t_0) = x_0, \quad y(t_0) = y_0, \tag{4.13}$$

has a unique solution that is valid for $-\infty < t < \infty$. (Note: this implies two orbits in the phase plane cannot intersect.) □

Finally, as illustrated by Example 4.6, we can write the form of the general solution to the system (4.5)–(4.6).

Theorem 4.10

(**General Solution**) If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ form a fundamental set of solutions of the linear system (4.5)–(4.6), then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t), \tag{4.14}$$

where c_1 and c_2 are arbitrary constants, is the general solution to (4.5)–(4.6). The general solution contains all possible solutions to the system for various choices of c_1 and c_2. □

This theorem gives the prescription for determining the general solution. If there are initial conditions, then they determine c_1 and c_2 and therefore the unique solution to the initial value problem, guaranteed by Theorem 4.9.

Remark 4.11

The method of elimination, that is, transforming a system into a single second-order equation, provides a method to find a fundamental set of solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ and therefore the general solution (4.14) to a linear system. See Example 4.6. However, a procedure using matrices, discussed in the next section, is a more elucidating approach to this problem. □

Example 4.12

The system

$$x' = y, \quad y' = 4x$$

can be transformed immediately into the second-order equation $x'' - 4x = 0$, which has solution

$$x(t) = c_1 e^{2t} + c_2 e^{-2t}.$$

Therefore, because $y = x'$,

$$y(t) = 2c_1 e^{2t} - 2c_2 e^{-2t}.$$

In terms of vectors, the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \end{pmatrix}.$$

The Wronskian of the two solutions is

$$W(t) = e^{2t}\left(-2e^{-2t}\right) - 2e^{2t}\left(e^{-2t}\right) = -4 \neq 0.$$

So the two vector solutions above form a fundamental set. If we impose and initial condition $x(0) = 0$, $y(0) = 1$, then

$$\mathbf{x}(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} c_1 + c_2 &= 0, \\ 2c_1 - 2c_2 &= 1, \end{aligned}$$

which has solution $c_1 = \frac{1}{4}$, $c_2 = -\frac{1}{4}$. Therefore the unique solution to the initial value problem is

$$\mathbf{x}(t) = \frac{1}{4} \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} - \frac{1}{4} \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \end{pmatrix}. \quad \square$$

There is another technique that applies to simple systems, leading to a formula for the orbits in terms of x and y, without the time-dependence.

Example 4.13

Consider again the system

$$\begin{aligned} x' &= y, \\ y' &= -4x. \end{aligned}$$

Here, x and y are functions of t. But if t is eliminated, then y can be written as a function of x. We can use the chain rule to get an expression for dy/dx, as follows.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},$$

so

$$\frac{dy}{dx} = \frac{dt/dt}{dx/dt} = \frac{-4x}{y}.$$

Therefore we obtained a first-order differential for y in terms of x, and we lost dependence on t. This amounts to dividing the two original equations. Now the equation is easily solved by separating variables to get

$$x^2 + \frac{1}{4}y^2 = C, \qquad C \text{ constant.}$$

These are concentric, elliptical orbits of the system expressed in terms of a single parameter C. A specific orbit can be determined from initial conditions. For example, if $x(0) = 1$ and $y(0) = 2$, then $C = 1^1 + \frac{1}{4}2^2 = 2$. Therefore the orbit is $x^2 + \frac{1}{4}y^2 = 2$. □

Remark 4.14

(**Division of Equations**) In general, if we have a system

$$x' = ax + by,$$
$$y' = cx + dy,$$

then, as in Example 4.13, we can divide the two equations to obtain

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by}.$$

This first-order equation is a differential equation for the orbits in terms of x and y, with time-dependence suppressed. We can find all the orbits by finding the general solution. However, this method works easily for only the simplest systems. □

Remark 4.15

(**Laplace transforms**) Another approach to solving linear systems is by Laplace transforms, the subject of Chapter 3. If $X(s)$ is the transform of $x(t)$, and $Y(s)$ is the transform of $y(t)$, then taking the transform of each equation in (4.5)–(4.6) gives

$$sX(s) - x(0) = aX(s) + bY(s), \tag{4.15}$$
$$sY(s) - y(0) = cX(s) + dY(s). \tag{4.16}$$

We can solve this system of two algebraic equations for $X(s)$ and $Y(s)$ and then return to the time domain using the inverse transform. This method is best for problems with piecewise continuous or point sources. □

As a final definition, a first-order **nonhomogeneous system** has the form

$$x' = ax + by + f_1(t),$$
$$y' = cx + dy + f_2(t),$$

where $f_1(t)$ and $f_2(t)$ are the two components of the vector source $\mathbf{f}(t)$. The method of elimination transforms this system into a nonhomogeneous second-order equation. We discuss nonhomogeneous equations in Section 4.5.

EXERCISES

1. Find and plot the following parametric curves in the phase plane for $-\infty < t < \infty$ by eliminating the time from the equations.

   a) $x = 3\sin 2\pi t, \;\; y = 4\cos 2\pi t.$
   b) $x = 3\cosh t, \;\; y = \sinh t.$
   c) $x = t^2 - 1, \;\; y = 2t.$
   d) $x = e^{-2t}, \;\; y = -2e^{-2t}.$

2. By dividing the following equations and integrating, find the equation of the orbits in terms of x and y for the following systems. Plot several representative orbits in the phase plane. Find the general solution of each system in component form and in vector form, indicating the fundamental set of solutions.

   a) $x' = -3y, \;\; y' = 2x.$
   b) $x' = -2y, \;\; y' = -4x.$
   c) $x' = -3x, \;\; y' = 2y.$
   d) $x' = 4y, \;\; y' = 2y.$

3. Use the method of elimination to find the general solution and write it as a linear combination of two vector solutions.

   a) $x' = x, \;\; y' = x + 2y.$
   b) $x' = x - y, \;\; y' = x + y.$
   c) $x' = x + 2y, \;\; y' = x.$
   d) $x' = -x - 2y, \;\; y' = 2x - y.$

4. Formulate a system of differential equations for the crop–soil model in Example 4.2 if there is a constant application k of the pesticide onto the soil.

4.2 Matrices and Linear Systems

The study of systems of differential equations is greatly facilitated by matrices. It provides a convenient and concise language and notation to express many of the ideas. Complicated formulas are simplified considerably in this framework, and matrix notation is independent of the number of equations. In the section we present a brief introduction to square matrices of dimension two. It does not represent a thorough treatment of matrix theory, but rather a limited discussion centered on ideas required to solve systems of differential equations. In Section 2 we formulate a system of differential equations in terms of matrices and use the formulation to study equilibrium solutions of the system.

4.2.1 Preliminaries from Algebra

Matrices and Vectors

A square array A of numbers having 2 rows and 2 columns is called a **square matrix** of size 2, or an 2×2 matrix; we say, "2 by 2 matrix". The number in the ith row and jth column is denoted by a_{ij}. A general 2×2 matrix has the form

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The numbers a_{ij} are called the *entries* in the matrix; the first subscript i denotes the row, and the second subscript j denotes the column. The **main diagonal** of a square matrix A is the set of elements a_{11}, a_{22}. We often write matrices using the brief notation $A = (a_{ij})$. A **vector x** is a *column list* of numbers x_1, x_2, called components,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Vectors are denoted by lowercase boldface letters such as $\mathbf{x}$, $\mathbf{y}$, etc., and matrices are denoted by capital letters such as A, B, etc. To minimize space in typesetting, we often write a vector $\mathbf{x}$ as a *transposed row list*, or $\mathbf{x} = (x_1, x_2)^{\mathrm{T}}$. The T means transpose, or turn the row into a column.

Two square matrices can be added entrywise. That is, if $A = (a_{ij})$ and $B = (b_{ij})$ are two matrices, then the **sum** $A+B$ is a matrix defined by $A+B = (a_{ij}+b_{ij})$. A matrix $A = (a_{ij})$ can be multiplied by a constant c by multiplying all the elements of A by the constant; in symbols this **scalar multiplication** is defined by $cA = (ca_{ij})$. Thus $-A = (-a_{ij})$. It is clear that $A + (-A) = 0$, where 0 is the **zero matrix** having all entries zero. *Subtraction* is defined by $A - B = A + (-B)$. Further, $A + 0 = A$. Addition is both commutative and associative. Therefore the arithmetic rules of addition for matrices are the same as the most of the usual rules for addition of real numbers.

Similar rules hold for addition of vectors and multiplication of column vectors by scalars. These are the definitions encountered in multivariable calculus where vectors are regarded as elements of $\mathbb{R}^2$. Vectors add componentwise, and multiplication of a vector by a scalar multiplies each component of that vector by that scalar.

Example 4.16

Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -2 \\ 7 & -4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

Then

$$A + B = \begin{pmatrix} 1 & 0 \\ 10 & -8 \end{pmatrix}, \quad -3B = \begin{pmatrix} 0 & 6 \\ -21 & 12 \end{pmatrix},$$

$$5\mathbf{x} = \begin{pmatrix} -20 \\ 30 \end{pmatrix}, \quad \mathbf{x}+2\mathbf{y} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}. \quad \square$$

The product of two square matrices is *not* found by multiplying entrywise. Rather, *matrix multiplication* is defined as follows. Let A and B be two matrices. Then the matrix AB is defined to be the matrix $C = (c_{ij})$ where the ij entry in the product C is found by taking the product (dot product, as with vectors) of the ith row of A and the jth column of B. In symbols, $AB = C$, where

$$c_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j = a_{i1}b_{1j} + a_{i2}b_{2j},$$

where $\mathbf{a}_i$ denotes the ith row of A, and $\mathbf{b}_j$ denotes the jth column of B. Generally, matrix multiplication is *not* commutative (i.e., $AB \neq BA$), so the order in which matrices are multiplied is important. However, the associative law $AB(C) = (AB)C$ does hold, so you can regroup products as you wish. The distributive law connecting addition and multiplication, $A(B+C) = AB+AC$, also holds. The powers of a square matrix are defined by $A^2 = AA$, $A^3 = AA^2$, and so on.

Example 4.17

Let

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 \\ 5 & 2 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 2\cdot 1 + 3\cdot 5 & 2\cdot 4 + 3\cdot 2 \\ -1\cdot 1 + 0\cdot 5 & -1\cdot 4 + 0\cdot 2 \end{pmatrix} = \begin{pmatrix} 17 & 14 \\ -1 & -4 \end{pmatrix}.$$

Also

$$\begin{aligned} A^2 &= \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2\cdot 2 + 3\cdot(-1) & 2\cdot 3 + 3\cdot 0 \\ -1\cdot 2 + 0\cdot(-1) & -1\cdot 3 + 0\cdot 0 \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ -2 & -3 \end{pmatrix}. \quad \square \end{aligned}$$

The special square matrix having ones on the main diagonal and zeros off the diagonal is called the **identity matrix** and is denoted by I. That is,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that if A is any square matrix and I is the identity matrix, then

$$AI = IA = A.$$

Therefore multiplication by the identity matrix does not change the result, a situation similar to multiplying real numbers by the unit number 1. If A is a given matrix and there exists a matrix B for which

$$AB = BA = I,$$

then B is called the **inverse** of A and we denote it by $B = A^{-1}$. If A^{-1} exists, we say A is a **nonsingular** matrix; otherwise it is called **singular**. Thus,

$$AA^{-1} = A^{-1}A = I.$$

We never write $1/A$ for the inverse of A.

Next we define multiplication of an 2×2 matrix A times a vector $\mathbf{x}$. The product $A\mathbf{x}$ with the *matrix on the left*, is defined to be a vector whose ith component is $\mathbf{a}_i \cdot \mathbf{x}$. In other words, the ith element in the list $A\mathbf{x}$ is found by taking the product of the ith row of A and the vector $\mathbf{x}$. Precisely,

$$A\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

For a numerical example take

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}.$$

Then

$$A\mathbf{x} = \begin{pmatrix} 2\cdot 5 + 3\cdot 7 \\ -1\cdot 5 + 0\cdot 7 \end{pmatrix} = \begin{pmatrix} 31 \\ -5 \end{pmatrix}. \quad \square$$

The product of a vector times a matrix, $\mathbf{x}A$, with the matrix A on the right, is not defined.

A useful number associated with a square matrix A is its determinant. The **determinant** of a square matrix A, denoted by $\det A$ is the number

$$\det A = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb,$$

For example,

$$\det \begin{pmatrix} 2 & 6 \\ -2 & 0 \end{pmatrix} = 2 \cdot 0 - (-2) \cdot 6 = 12. \quad \square$$

Using the determinant we can give a simple formula for the inverse of a matrix A. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and suppose $\det A \neq 0$. Then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{4.17}$$

So the inverse of a 2×2 matrix is found by interchanging the main diagonal elements, putting minus signs on the off-diagonal elements, and dividing each entry by the determinant.

Example 4.18

If

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix},$$

then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix} = \frac{1}{-5} \begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & \frac{2}{5} \\ \frac{4}{5} & -\frac{1}{5} \end{pmatrix}.$$

The reader can easily check that $AA^{-1} = I$. $\square$

Equation (4.17) shows that the inverse matrix exists if, and only if, the determinant is nonzero. Formally,

$$A^{-1} \text{ exists if, and only if, } \det A \neq 0.$$

This is a major result in matrix theory, and it is a convenient test for invertibility of a matrix.

Systems of Algebraic Equations

Matrices were developed to represent and study linear algebraic systems (m linear algebraic equations in n unknowns) in a concise way. Here we are taking $m = n = 2$. Consider two equations in two unknowns x_1, x_2 given in standard form by

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2.$$

Using matrix notation we can write this as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

In more concise notation this system is simply

$$A\mathbf{x} = \mathbf{b}, \tag{4.18}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

A is called the **coefficient matrix**, $\mathbf{x}$ is a vector containing the two unknowns, and $\mathbf{b}$ is a vector representing the right side of the equations. If $\mathbf{b} = \mathbf{0}$, the zero vector, then the system (4.18) is called **homogeneous**. Otherwise it is called **nonhomogeneous**. Geometrically, in a two-dimensional system each equation represents a line in the plane. A solution vector $\mathbf{x}$ is represented by a point that lies on both lines. There is a unique solution when both lines intersect at a single point; there are infinitely many solutions when both lines coincide; there is no solution if the lines are parallel and different. When $\mathbf{b} = \mathbf{0}$ the two lines pass through the origin and $\mathbf{x} = \mathbf{0}$ is always a solution to the homogeneous system; or, there are infinitely many solutions when the lines coincide. The following theorem tells us when a linear system $A\mathbf{x} = \mathbf{b}$ of 2 equations in 2 unknowns is solvable. It is a important result that is applied often in this chapter.

Theorem 4.19

(1) If $\det A \neq 0$, then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution given by $\mathbf{x} = A^{-1}\mathbf{b}$; the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

(2) If $\det A = 0$, then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions; the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ either has no solution or has infinitely many solutions. □

It is easy to show the first part of the theorem, when A is nonsingular, or $\det \neq 0$, using the machinery of matrix notation. Multiplying both sides of $A\mathbf{x} = \mathbf{b}$ on the left by A^{-1} gives

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b}, \\ I\mathbf{x} &= A^{-1}\mathbf{b}, \\ \mathbf{x} &= A^{-1}\mathbf{b}, \end{aligned}$$

which is the unique solution. In the case A is singular, or $\det A = 0$, we can appeal to a geometric argument as follows. The two lines represented by the two equations must be parallel (you should show this), and therefore they either coincide or they do not, giving either infinitely many solutions or no solution.

Example 4.20

Consider the homogeneous linear system

$$\begin{pmatrix} 4 & 1 \\ 8 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The coefficient matrix has determinant zero, so there will be infinitely many solutions. The two equations represented by the system are

$$4x_1 + x_2 = 0, \quad 8x_1 + 2x_2 = 0,$$

which are clearly not independent because one is a multiple of the other. Therefore we need only consider one of the equations, say $4x_1 + x_2 = 0$. With one equation in two unknowns we are free to pick a value for one of the variables and solve for the other. Let $x_1 = 1$; then $x_2 = -4$ and we get a single solution $\mathbf{x} = (1, -4)^{\mathrm{T}}$. More generally, if we choose $x_1 = \alpha$, where α is any real parameter, then $x_2 = -4\alpha$. Therefore all solutions are given by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ -4\alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \quad \alpha \in \mathbb{R}.$$

Thus all solutions are multiples of $(1, -4)^{\mathrm{T}}$, and the solution set lies along the straight line through the origin with slope -4. Geometrically, the two equations represent two lines in the plane that coincide. □

The set of all solutions to a homogeneous system $A\mathbf{x} = \mathbf{0}$ is called the **nullspace** of A. The nullspace consists of a single point $\mathbf{x} = \mathbf{0}$ when A is nonsingular ($\det A \neq 0$), and it is a line passing through the origin in the case where A is singular ($\det A = 0$). For example, the straight line $x_2 = -4x_1$ is the nullspace of the matrix A in the last example.

There is a simple, easily verified, algorithm to solve nonhomogeneous systems of two equations in two unknowns, called Cramer's rule.

Theorem 4.21

(**Cramer's Rule**) The solution to the nonhomogeneous linear system

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

is

$$x_1 = \frac{\det \begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}}{\det A}, \quad x_2 = \frac{\det \begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}}{\det A}. \quad \square$$

Example 4.22

Solve the linear system

$$\begin{aligned} 2x - 3y &= 4, \\ 4x - y &= -1. \end{aligned}$$

The coefficient matrix is

$$\begin{pmatrix} 2 & -3 \\ 4 & -1 \end{pmatrix},$$

which has determinant $\det A = 10$. Therefore, using Cramer's rule,

$$x = \frac{\det \begin{pmatrix} 4 & -3 \\ -1 & -1 \end{pmatrix}}{10} = \frac{-7}{10}, \quad y = \frac{\det \begin{pmatrix} 2 & 4 \\ 4 & -1 \end{pmatrix}}{10} = \frac{-18}{10}. \quad \square$$

Next we introduce the notion of linear independence of a set of vectors in the plane. Two vectors

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

are **linearly independent** if one of them is not a multiple of the other. We can express this statement equivalently as follows. The set $\mathbf{v}$, $\mathbf{w}$ of vectors is linearly independent if the only linear combination of the vectors equaling the zero vector is the zero combination; that is, if

$$c_1\mathbf{v} + c_2\mathbf{w} = \mathbf{0},$$

then $c_1 = c_2 = 0$. This is easily rewritten in matrix form as a homogeneous system for c_1 and c_2:

$$\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If

$$\det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \neq 0, \tag{4.19}$$

then the only solution is the trivial solution $c_1 = c_2 = 0$, implying $\mathbf{v},\ \mathbf{w}$ is a linearly independent set. If two vectors are not linearly independent then one is a multiple of the other and the determinant in (4.19) is zero; in this case we say the vectors are **linearly dependent**.

EXERCISES

1. Let
$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 3 & 7 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}.$$
Find $A + B$, $B - 4A$, AB, BA, A^2, $B\mathbf{x}$, $AB\mathbf{x}$, A^{-1}, $\det B$, B^3, AI, and B^{-1}.

2. Let A be the matrix in Exercise 1.
   a) With $\mathbf{b} = (2, 1)^{\mathrm{T}}$, solve the system $A\mathbf{x} = \mathbf{b}$ for unknown $\mathbf{x}$ using the inverse matrix A^{-1}.
   b) Solve the same system by Cramer's rule.
   c) Illustrate the solution geometrically in the xy plane.

3. Let A be the matrix in Exercise 1.
   a) Find all values of the parameter λ that satisfy the equation $\det(A - \lambda I) = 0$.
   b) For each of the values of λ determine the solution of the system $(A - \lambda I)\mathbf{x} = 0$.

4. Repeat Exercises 2 and 3 for the matrix B in Exercise 1.

5. Let
$$A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$
Compute $\det A$. Does A^{-1} exist? Find all solutions to $A\mathbf{x} = \mathbf{0}$ and plot the solution set in the plane. What is the nullspace of A?

6. Write each system in matrix form, and for each system determine all values m for which the system has no solution, a unique solution, or infinitely many solutions.

a) $2x + 3y = m, \quad -6x - 9y = 5.$

b) $2x + my = 6, \quad x + y = 2.$

c) $mx - 2y = 0, \quad 2x + 4y = 0.$

d) $mx + 3y = m, \quad 3x - my = 1.$

7. If a square matrix A has all zeros either below its main diagonal or above its main diagonal, show that $\det A$ equals the product of the elements on the main diagonal.

8. Use the definition of linear independence to show that the vectors $(2, -3)^{\mathrm{T}}$ and $(-4, 8)^{\mathrm{T}}$ are linearly independent.

4.2.2 Differential Equations and Equilibria

A two-dimensional linear system of differential equations

$$x' = ax + by, \tag{4.20}$$

$$y' = cx + dy, \tag{4.21}$$

where a, b, c, and d are constants, can be written compactly using vectors and matrices. Denoting

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the system can be written

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

or

$$\mathbf{x}'(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x}(\mathbf{t}).$$

Simply stated,

$$\mathbf{x}' = A\mathbf{x}, \tag{4.22}$$

where we suppressed the understood dependence of x on t. Because the right sides of the equations (4.20)–(4.21) do not contain time t explicitly, we say the system is **autonomous**.

Equilibria and Stability

Just as for first-order autonomous equations studied in Chapter 1, the equilibrium, or constant, solutions and their stability are key concepts in the study of linear and nonlinear systems of differential equations. An **equilibrium solution** of $\mathbf{x}' = A\mathbf{x}$, or (4.20)–(4.21), is a *constant* vector solution $\mathbf{x}(t) = \mathbf{x}^*$ for which

$$A\mathbf{x}^* = \mathbf{0}.$$

This is clearly equivalent to

$$ax^* + by^* = 0, \quad cx^* + dy^* = 0$$

where $\mathbf{x}^* = (x^*, y^*)^{\mathrm{T}}$. The component representation of an equilibrium solution is two constant functions $x(t) = x^*$, $y(t) = y^*$ plotted against t. Because an equilibrium solution plots as a point in the phase plane, we refer to it as a **critical point** and use the notation $\mathbf{x}^* = (x^*, y^*)$, without the transpose. Obviously, the critical points, or equilibria, are solutions of

$$ax + by = 0, \quad cx + dy = 0,$$

which are at the intersections of the two straight lines $ax+by = 0$ and $cx+dy = 0$. The theorem summarizes what we know.

Theorem 4.23

The linear system $\mathbf{x}' = A\mathbf{x}$ has:

1. A unique equilibrium at $(0, 0)$ if $\det A \neq 0$.
2. A straight line of equilibria if $\det A = 0$.

If the origin $(0, 0)$ is the only critical point, we say the origin is an **isolated critical point** because there is a neighborhood of $(0, 0)$ that contains no other critical point. Otherwise it is called **nonisolated**.

Remark 4.24

The system

$$\mathbf{x}' = A\mathbf{x} + \mathbf{b}, \quad \det A \neq \mathbf{0}, \tag{4.23}$$

where $\mathbf{b} = (b_1, b_2)^{\mathrm{T}}$ is a *constant* vector, is also an autonomous system, but it is nonhomogeneous. A critical point $\mathbf{x}^*$ for (4.23) is found by setting the right side equal to zero, or $A\mathbf{x}^* + \mathbf{b} = \mathbf{0}$. Hence,

$$\mathbf{x}^* = -A^{-1}\mathbf{b}. \tag{4.24}$$

Nonhomogeneous systems (4.23) can be reduced to a homogeneous system of the form (4.22) by translating the critical point to the origin as follows. Let

$$\mathbf{y} = \mathbf{x} - \mathbf{x}^*. \tag{4.25}$$

Then

$$\mathbf{y}' = \mathbf{x}' = A\mathbf{x} + \mathbf{b} = A(\mathbf{y} + \mathbf{x}^*) + \mathbf{b} = A\mathbf{y} + A\mathbf{x}^* + \mathbf{b} = A\mathbf{y},$$

where we used (4.24). Therefore we can solve (4.23) by solving $\mathbf{y}' = A\mathbf{y}$ for $\mathbf{y}$ and then using (4.25) to get $\mathbf{x}$. Note that if $\det A = 0$, then A^{-1} does not exist and there may be no equilibria to (4.23). □

Suppose a homogeneous system has an isolated critical point at $(0, 0)$. Intuitively, the equilibrium is *asymptotically stable* if there is a small neighborhood (open circle) about $(0, 0)$ such that every orbit $\mathbf{x}(t)$ that starts in the circle at time $t = t_0$ converges to $(0, 0)$ as $t \to \infty$. The origin is *stable* if nearby orbits stay nearby for all future time; they do not need to converge to the origin. The origin is *unstable* if it is not stable. Nature selects out the stable equilibria, so an equilibrium must be stable to be physically meaningful; in real systems there are always small changes from equilibrium.

For interested readers, a more technical characterization is:

Definition 4.25

1. The isolated equilibrium $\mathbf{x}^* = \mathbf{0}$ is **asymptotically stable** if, and only if, there is a circle $C_\epsilon : x^2 + y^2 < \epsilon$ centered at the origin such that every orbit which begins in the circle C_ϵ at any time $t = t_0$ approaches the origin as $t \to +\infty$.
2. The isolated equilibrium $\mathbf{x}^* = \mathbf{0}$ is **stable** if, and only if, for every circle $C_\epsilon : x^2 + y^2 < \epsilon$ centered at the origin, there is a circle C_δ, $\delta < \epsilon$, inside inside C_ϵ such that every orbit which begins in the circle C_δ at any time $t = t_0$ remains in the circle C_ϵ for all times $t > t_0$.
3. If the isolated equilibrium is not stable, then it is called **unstable**. □

It is clear that asymptotically stable implies stable. An asymptotically stable equilibrium, where orbits approach the origin, is also called an **attractor** or **sink**. Some unstable equilibria are called **repellers**, or **sources** when they exit every neighborhood. If the orbits form set of concentric ellipses around the origin, it is stable; some texts use the term *neutrally stable*. We use these terms in the sequel to describe the types of equilibria that can occur in a system.

EXERCISES

1. Write each of the following systems in matrix format and identify the coefficient matrix.

   a) $x' = -2x - 3y,\ y' = -x + 4y.$
   b) $x' = -3y,\ y' = -2x + y.$
   c) $x' = -2x,\ y' = x.$
   d) $x' = -2x - y,\ y' = -4y.$
   e) $x' = x - 2y,\ y' = -2x + 4y.$
   f) $x = -6y,\ y' = 6y.$

2. Determine the critical points for the systems in Exercise 1.

3. Find the critical points for each nonhomogeneous system and then transform the system into a homogeneous system.

   a) $x' = 2x + 3y,\ \ y' = -x - 14.$
   b) $x' = -x+3y-6,\ \ y' = x+2y-1.$

4.3 The Eigenvalue Problem

Next we introduce matrix methods to solve the two-dimensional system

$$\mathbf{x}' = A\mathbf{x}. \tag{4.26}$$

Referring to (4.20) and (4.21), we observe that a solution $x(t),\ y(t)$ must have the property that $x(t),\ y(t)$ and their derivatives have the same form if all terms are to cancel. Exponential solutions satisfy this criterion. Therefore, we attempt to find a solution of the form

$$\mathbf{x} = \mathbf{v}e^{\lambda t}, \tag{4.27}$$

where λ is a constant and $\mathbf{v}$ is a nonzero constant vector, both to be determined. Substituting $\mathbf{x} = \mathbf{v}e^{\lambda t}$ and $\mathbf{x}'=\lambda\mathbf{v}e^{\lambda t}$ into (4.26) gives

$$\lambda\mathbf{v}e^{\lambda t} = A(\mathbf{v}e^{\lambda t}),$$

or

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{4.28}$$

Therefore, if some λ and $\mathbf{v}$ can be found that satisfy (4.28), then we have determined a solution of the form (4.27). Equation (4.28) is a famous, important problem in mathematics called the **algebraic eigenvalue problem**. The following remark summarizes the key idea in solving differential equations.

Remark 4.26

A nonzero constant vector $\mathbf{v}$ is an **eigenvector** of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some λ. The constant λ is called an **eigenvalue** of A. The pair $\lambda, \mathbf{v}$ is called an **eigenpair**. Every eigenpair of A gives a solution $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$ of $\mathbf{x}' = A\mathbf{x}$. □

Geometrically we think of the eigenvalue problem (4.28) like this: the matrix A represents a transformation that acts on vectors to produce vectors; that is, a vector $\mathbf{v}$ gets transformed to a vector $A\mathbf{v}$. An eigenvector of A is a special vector that is transformed to a multiple (λ) of itself; that is, $A\mathbf{v} = \lambda\mathbf{v}$.

In summary, we have reduced the problem of finding solutions to a system of differential equations to the problem of finding solutions of an algebraic problem—every eigenpair gives a solution to the system (4.26). In the remainder of this section we focus on solving the eigenvalue problem. In the next section we classify all the types of solutions the system $\mathbf{x}' = A\mathbf{x}$, depending upon the eigenvalues of A.

Solving the eigenvalue problem. We rewrite (4.28) as a homogeneous linear system

$$(A-\lambda I)\mathbf{v} = \mathbf{0}. \tag{4.29}$$

This system has nontrivial solutions if the determinant of the coefficient matrix is zero, or

$$\det(A - \lambda I) = 0. \tag{4.30}$$

Written out explicitly, this system (4.29) has the form

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the coefficient matrix $A-\lambda I$ is the matrix A with λ subtracted from the diagonal elements. Equation (4.30) is, explicitly,

$$\det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - cb = 0,$$

or equivalently,

$$\lambda^2 - (a+b)\lambda + (ad - bc) = 0. \tag{4.31}$$

Note that this equation is exactly the equation (4.7) that we obtained by the method of elimination in Section 4.2. This equation can be memorized easily if it is written

$$\lambda^2 - (\mathrm{tr}A)\lambda + \det A = 0, \tag{4.32}$$

where $\mathrm{tr}A = a + d$ is called the **trace** of A, defined to be the sum of the diagonal elements of A. Equation (4.31), or (4.32), is called the **characteristic**

equation associated with the matrix A, and it is a quadratic equation in λ. Its roots, found by factoring or using the quadratic formula, are the two eigenvalues of A. The eigenvalues may be real and unequal, real and equal, or complex conjugates. Once the eigenvalues are computed, we can substitute them in turn into the system (4.29) to determine corresponding eigenvectors $\mathbf{v}$. □

SUMMARY. To solve the eigenvalue problem $A\mathbf{v} = \lambda\mathbf{v}$:

(1) Solve the characteristic equation $\det(A - \lambda I) = 0$, or

$$\lambda^2 - (\mathrm{tr}A)\lambda + \det A = 0,$$

to determine the eigenvalues λ.

(2) Successively substitute each eigenvalue λ into the homogeneous system

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

to find the eigenvectors $\mathbf{v}$ that correspond to each λ.

Before presenting examples, we state two important facts.

(1) *Any constant multiple of an eigenvector corresponding to an eigenvalue λ is again an eigenvector for the same eigenvalue.* This follows from the calculation

$$A(c\mathbf{v}) = cA\mathbf{v} = c(\lambda\mathbf{v}) = \lambda(c\mathbf{v}).$$

Thus, an eigenvector corresponding to a given eigenvalue is unique up to a constant multiple.

(2) *If the eigenvalues are unequal, i.e., $\lambda_1 \neq \lambda_2$, then the corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent.* To prove this, assume not. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are proportional. Hence there is a *nonzero* number c such that

$$\mathbf{v}_2 = c\mathbf{v}_1. \tag{4.33}$$

Multiplying on the left by A, we get

$$A\mathbf{v}_2 = cA\mathbf{v}_1,$$

or

$$\lambda_2\mathbf{v}_2 = c\lambda_1\mathbf{v}_1.$$

Next, multiply (4.33) by λ_2 to get

$$\lambda_2\mathbf{v}_2 = c\lambda_2\mathbf{v}_1.$$

Subtracting the last two equations, we obtain $\mathbf{0} = c(\lambda_1 - \lambda_2)\mathbf{v}_1$. This statement is a contradiction because every term on the right is nonzero. Therefore, $\mathbf{v}_1$ and $\mathbf{v}_2$ are not proportional and must be independent.

Example 4.27

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}.$$

To find the eigenvalues we write down the characteristic equation $\det(A - \lambda I) = 0$, or

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = 0,$$

or

$$(1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3) = 0.$$

Thus the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$. Note that the characteristic equation is $\lambda^2 - (\mathrm{tr}A)\lambda + \det A = 0$ where $\mathrm{tr}A = 2$ and $\det A = -3$.

Now we find eigenvectors associated with each eigenvalue by successively substituting the eigenvalue into $(A - \lambda I)\mathbf{v} = \mathbf{0}$, or

$$\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = -1$ the system becomes

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The two equations represented are $2v_1 + v_2 = 0$ and $4v_1 + 2v_2 = 0$, which are the same. This is expected because the coefficient determinant is zero; that is how we obtain eigenvalues. Therefore we need to consider only one of the equations, say $2v_1 + v_2 = 0$. The solution is $v_2 = -2v_1$, where v_1 is arbitrary. Therefore take $v_1 = 1$, and $v_2 = -2$, giving the eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

So

$$\lambda_1 = -1, \ \ \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

is an eigenpair. Recall, any multiple of an eigenvector is again an eigenvector; but we only need to choose a representative one.

Next we take $\lambda_2 = 3$. Then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ becomes

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In the same manner as above we need to take only one of the two equations, say $-2v_1 + v_2 = 0$. Then choosing the solution $v_1 = 1$, $v_2 = 2$, we obtain the second eigenpair

$$\lambda_2 = 3, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \quad \square$$

The same process works for complex eigenvalues. The complex eigenpairs are always complex conjugates.

Example 4.28

(**Complex eigenvalues**) Because eigenvalues are solutions to a quadratic equation, they may be complex conjugate numbers. The resulting eigenvectors will *always* be two complex conjugate vectors. Let

$$A = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}.$$

The characteristic equation is

$$\det(A - \lambda I) = \det \begin{pmatrix} -2-\lambda & -3 \\ 3 & -2-\lambda \end{pmatrix} = \lambda^2 + 4\lambda + 13 = 0.$$

Using the quadratic formula, we get eigenvalues $\lambda = -2 \pm 3i$. Next we find eigenvectors associated with $\lambda = -2 + 3i$ by solving the system $(A - \lambda I)\mathbf{v} = \mathbf{0}$, or

$$\begin{pmatrix} -2-(-2+3i) & -3 \\ 3 & -2-(-2+3i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or, in component form,

$$-3iv_1 - 3v_2 = 0,$$
$$3v_1 - 3iv_2 = 0.$$

These equations are proportional (the second is i times the first), as expected, so we need to consider only one, say $3v_1 - 3iv_2 = 0$. Taking $v_2 = i$ we get $v_1 = -1$. So the eigenvectors corresponding to $\lambda = -2 + 3i$ are all multiples of

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ i \end{pmatrix}.$$

The eigenvectors associated with $\lambda = -2 - 3i$ are all multiples of

$$\mathbf{v}_2 = \begin{pmatrix} -1 \\ -i \end{pmatrix},$$

which is the complex conjugate of the vector $\mathbf{v}_1$. The eigenpairs are

$$-2 \pm 3i, \quad \begin{pmatrix} -1 \\ \pm i \end{pmatrix}. \quad \square$$

EXERCISES

1. Find the eigenvalues and eigenvectors of the following matrices.

a) $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$. f) $\begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}$. k) $\begin{pmatrix} -4 & \frac{1}{4} \\ 4 & -4 \end{pmatrix}$.

b) $\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$. g) $\begin{pmatrix} \frac{1}{2} & -\frac{5}{4} \\ 2 & -\frac{1}{2} \end{pmatrix}$. l) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

c) $\begin{pmatrix} 0 & -2 \\ 3 & 0 \end{pmatrix}$. h) $\begin{pmatrix} -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{pmatrix}$. m) $\begin{pmatrix} -7 & 6 \\ 6 & 2 \end{pmatrix}$.

d) $\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$. i) $\begin{pmatrix} -2 & -3 \\ 2 & 4 \end{pmatrix}$. n) $\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$.

e) $\begin{pmatrix} -1 & 4 \\ \frac{1}{2} & -2 \end{pmatrix}$. j) $\begin{pmatrix} -2 & 0 \\ 2 & -4 \end{pmatrix}$. o) $\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$.

2. Find the eigenvalues of the following matrices and describe their dependence on β.

a) $\begin{pmatrix} 2 & \beta \\ -1 & 0 \end{pmatrix}$. b) $\begin{pmatrix} 1 & -\beta \\ 2\beta & 3 \end{pmatrix}$. c) $\begin{pmatrix} -1 & -1 \\ -\beta & -1 \end{pmatrix}$.

3. Show that $\det A = 0$ if, and only if, $\lambda = 0$ is an eigenvalue.

4. If λ is a nonzero eigenvalue of a matrix A, show that λ^{-1} is an eigenvalue of the inverse matrix A^{-1}.

5. If λ is an eigenvalue of a matrix A, show that λ^n is an eigenvalue of the matrix A^n, where $n \geq 2$ is an integer.

6. Can $\lambda = 0$ be an eigenvalue of a matrix A? Give an example.

7. Let $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$. If $(1, -2)^{\mathrm{T}}$ is an eigenvector, what is the eigenvalue?

4.4 Solving Linear Systems

We learned to solve the eigenvalue problem and now we are ready to classify and solve linear systems, depending on their eigenvalues. There are multiple cases because the eigenvalues of the matrix A can be real and unequal, real and equal, complex, or purely imaginary. The key idea is that each eigenpair λ, $\mathbf{v}$ leads to a solution $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$. In carrying out this process we also relate the structure of the solution to the orbits they represent in the phase plane. The structure of the orbits also elucidates the stability properties of the equilibria of the system.

4.4.1 Real Unequal Eigenvalues

In this section we begin analyzing the various solution forms of a linear system $\mathbf{x}' = A\mathbf{x}$, depending on the nature of the eigenvalues of the coefficient matrix A.

If the two eigenvalues of the system

$$\mathbf{x}' = A\mathbf{x}$$

are **real and unequal**, say λ_1 and λ_2, then corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent and we obtain two independent solutions $\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}$ and $\mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}$. The **general solution** of the system is then a linear combination of these two independent solutions,

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}, \tag{4.34}$$

where c_1 and c_2 are arbitrary constants.

To understand the orbital structure in the phase plane, there are three subcases to consider:

(i) λ_1 and λ_2 have opposite signs.

(ii) λ_1 and λ_2 are both negative and unequal.

(iii) λ_1 and λ_2 are both positive and unequal.

Example 4.29

(**Eigenvalues with opposite signs**) We start with a simple representative equation where eigenvalues have opposite signs, namely the system

$$\mathbf{x}' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}.$$

The two eigenpairs of the coefficient matrix A are very easily found to be

$$\lambda_1 = -1, \;\; \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = 1, \;\; \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t},$$

which is a linear combination of the two basic solutions

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}, \quad \mathbf{x}_2(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}.$$

The first basic solution $\mathbf{x}_1(t)$ associated with the negative eigenvalue represents an orbit moving toward the origin along the positive x axis. The solution $-\mathbf{x}_1(t)$ is an oppositely opposed orbit moving toward the origin along the negative x axis. Similarly, the second basic solution $\mathbf{x}_2(t)$ associated with the positive eigenvalue represents an orbit moving away from the origin on the positive y axis, and $-\mathbf{x}_2(t)$ is an oppositely oppose orbit moving away from the origin on the negative y axis. See Figure 4.5. What about the other oribits? In this case we can find them exactly by dividing the differential equations to get $y'/y = -x'/x$ and integrating to find $xy = C$, where C is a constant. (Alternately, you could write the solution in component form $x = c_1 e^{-t}$, $y = c_2 e^{t}$ and note the same result.) These orbits are hyperbolas $y = C/x$, also shown in Figure 4.5. This type of orbital structure is called a **saddle structure** and the origin is called a **saddle point**. A saddle point is clearly an unstable equilibrium. *This example is representative of all systems with real unequal eigenvalues.*

Example 4.30

Consider the system of differential equations

$$x' = x + y,$$
$$y' = 4x + y,$$

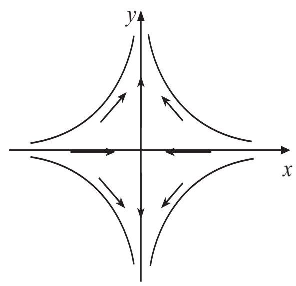

Figure 4.5 The phase plane diagram for Example 4.29 showing a saddle structure near the origin. The origin is called a saddle point and it is unstable.

or

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}.$$

The coefficient matrix for the system is the same as in Example 4.27, which has eigenpairs

$$\lambda_1 = -1, \ \ \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = 3, \ \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Therefore, a fundamental set of solutions is

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \qquad \text{and} \qquad \mathbf{x}_2(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}.$$

Therefore the general solution to the system is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \tag{4.35}$$

where c_1 and c_2 are arbitrary constants.

Again, this is a saddle structure. First consider $\mathbf{x}_1(t)$. It is a scalar (function) e^{-t} times the eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Therefore the solution is a straight line of slope $-2/1 = -2$ along the direction of the eigenvector. The parametric equations of this solution are

$$x(t) = e^{-t}, \ \ y(t) = -2e^{-t}, \quad -\infty < t < +\infty.$$

As $t \to +\infty$, the solution approaches the origin $(0, 0)$; as $t \to -\infty$, the solution approaches infinity. In summary, as t ranges forward in time from $-\infty$ to $+\infty$, the orbit corresponding to the solution is traced from infinity to $(0, 0)$ along the straight line in the 4th quadrant. See Figure 4.6. Notice that

$$\frac{x(t)}{y(t)} = \frac{-2e^{-t}}{e^{-t}} = -2 \text{ or } y(t) = -2x(t),$$

confirming the solution is on the line $y = -2x$. We call such a straight line solution a *linear orbit*. Corresponding to $\mathbf{x}_1(t)$ is the oppositely opposed orbit $-\mathbf{x}_1(t)$ in the second quadrant. Figure 4.6. In the same way, the solution

$$\mathbf{x}_2(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

represents a outgoing ray in the first quadrant going from $(0, 0)$ to infinity as t varies from $-\infty$ to $+\infty$ along the line $y = 2x$. Again, $-\mathbf{x}_2(t)$ is an outgoing, oppositely opposed ray in the third quadrant.

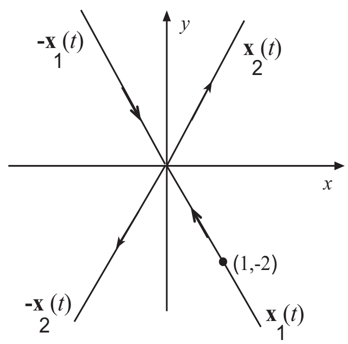

Figure 4.6 The linear orbits for Example 4.30: two pairs of opposing rays, one pair coming toward the origin (associated with the negative eigenvalue), and one pair leaving the origin (associated with the positive eigenvalue).

What about the other oribits? In (4.35) note that as $t \to +\infty$ we have $e^{-t} \to 0$ and

$$\mathbf{x}(t) \to c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} \text{ as } t \to +\infty.$$

Therefore all the orbits approach the line $y = 2x$, which is in the direction of the eigenvector associated with the positive eigenvalue. In the same way, as $t \to -\infty$, *backward* in time, $e^{3t} \to 0$ and

$$\mathbf{x}(t) \to c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \text{ as } t \to -\infty.$$

Therefore the orbits approach the straight line $y = -2x$ backward in time, which is in the direction of the positive eigenvalue. Thus, the orbits have a hyperbolic shape as they come toward the origin and veer away, approaching the linear orbits associated with the positive eigenvalue. Again we get a saddle structure as shown in Figure 4.7. In this case we also call the linear orbits **separatrices** because they separate different types of orbits in the phase plane. □

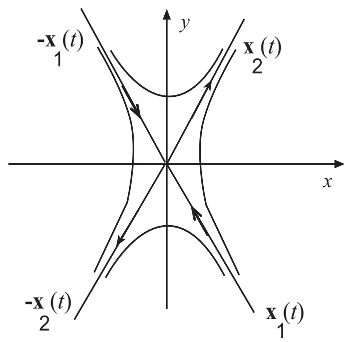

Figure 4.7 The phase plane diagram in Example 4.30 showing the linear orbits and the hyperbolic-like orbits that approach the linear orbits associated with the positive eigenvalue. Again, the origin is a saddle point.In the case of a saddle point, the linear orbits are called *separatrices*.

Finding the linear orbits of any system with two real eigenvalues is an essential tool in constructing the phase diagram. We highlight this important geometrical interpretation of a solution $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$ to a linear system.

Remark 4.31

If λ is a real eigenvalue of the linear system $\mathbf{x}' = A\mathbf{x}$ with corresponding real eigenvector $\mathbf{v}$, then the solution $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$ of the linear system is called a **linear orbit**. In the phase plane it plots as a ray along a straight line in the direction of the eigenvector $\mathbf{v}$, either entering the origin (if $\lambda < 0$) or exiting the origin (if $\lambda > 0$.) Further, $\mathbf{x}(t) = -\mathbf{v}e^{\lambda t}$ is a linear orbit in the opposite quadrant with the same properties. Therefore, an eigenpair gives rise to two opposing rays, both either approaching the origin or both exiting the origin. Hence, *every nonzero real eigenvalue leads to two opposing linear orbits, or rays, in phase space.*

In summary, each independent solution, $\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}$ and $\mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}$, represents a **linear orbit**, or **ray**, in the phase plane in the directions of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Their negatives $-\mathbf{x}_1(t)$ and $-\mathbf{x}_2(t)$ represent opposing linear orbits or rays in the opposite quadrants. Opposing orbits enter the origin for negative eigenvalues and leave the origin for positive eigenvalues. □

Example 4.32

(**Both eigenvalues negative**) Consider a representative system consisting of the decoupled equations

$$x' = -x,$$
$$y' = -2y.$$

Because the equations are *decoupled*, we can solve them directly to get

$$x(t) = c_1 e^{-t}, \quad y(t) = c_2 e^{-2t}. \tag{4.36}$$

In vector form the solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}.$$

It is clear that the eigenvalues are $\lambda = -1, -2$ with corresponding eigenvectors $(1,0)^{\mathrm{T}}$ and $(0,1)^{\mathrm{T}}$. Each basic solution represents a pair of opposing linear orbits along the x and y axes, respectively. Both sets of linear orbits approach the origin as $t \to +\infty$ because both eigenvalues are negative. Therefore, *all* solutions decay and their orbits approach the origin. Here, we can find the shapes of the orbits by eliminating the time parameter t. Square the first equation in (4.36) and then divide the two equations to get

$$y = \frac{c_2}{c_1^2}\, x^2 = Cx^2,$$

which are parabolas. If $C > 0$ they are concave up, and if $C < 0$ they are concave down. When $C = 0$ we get $y = 0$, or the the positive and negative x axes; these are the linear orbits. The positive and negative y axes are also linear orbits found by taking $c_1 = 0$. See Figure 4.8. In addition, the direction field has components $(-x, -2y)$; therefore, in the first quadrant curves approach the origin ($x' < 0$, $y' < 0$), and similarly for the other three quadrants. This type of structure is called a **nodal structure**, and the origin is an **asymptotically stable node**. This example is characteristic of systems with negative, unequal eigenvalues. □

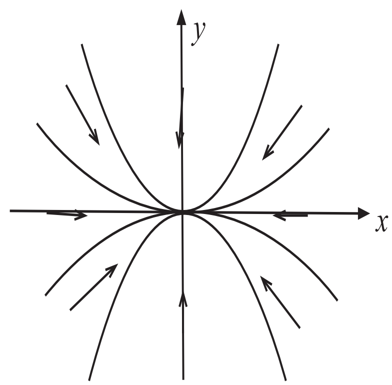

Figure 4.8 Orbits for the system $x' = -x$, $y' = -2y$, giving an asymptotically stable node at the origin. The direction field in each quadrant is shown as well.

Example 4.33

Consider a slightly more complicated system

$$x' = -2x + 2y,$$
$$y' = 2x - 5y.$$

The eigenpairs are

$$-1, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad -6, \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The component form of the general solution is

$$x(t) = 2c_1e^{-t} + c_2e^{-6t},$$
$$y(t) = c_1e^{-t} - 2c_2e^{-6t},$$

where c_1 and c_2 are arbitrary constants. The eigenvalues are -1 and -6, so the origin is an asymptotically stable node. Let's inquire about the phase diagram. How do the orbits enter the origin as $t \to +\infty$? For very large t, the e^{-6t} term decays much faster than the e^{-t} term. Therefore, in the limit the general solution approaches

$$x(t) \approx 2c_1e^{-t}, \quad \text{as} \quad t \to +\infty,$$
$$y(t) \approx c_1e^{-t}, \quad \text{as} \quad t \to +\infty,$$

which has the same direction as the is the linear orbit associated with the eigenvector,

$$x(t) = 2c_1e^{-t},$$
$$y(t) = c_1e^{-t}.$$

Thus, *all* orbits approach the origin tangent to the line $y = x/2$ as $t \to +\infty$.

As $t \to -\infty$ (backward in time) the term e^{-6t} dominates the term e^{-t} as t gets large negatively. Consequently, on any orbit

$$x(t) \approx c_2 e^{-6}, \quad \text{as} \quad t \to -\infty,$$
$$y(t) \approx -2c_2 e^{-6t}, \quad \text{as} \quad t \to -\infty.$$

Therefore the slope of all limiting orbits at $t = -\infty$ coincides with the linear orbits associated with the eigenvalue $\lambda = -6$. In summary, for large negative t, all the orbits become parallel to the line $y = -2x$. Figure 4.9 shows the phase diagram. □

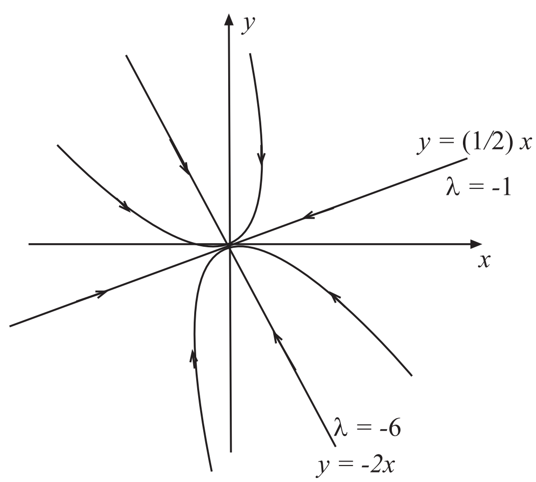

Figure 4.9 The linear orbits and the behavior of orbits as $t \to +\infty$ and as $t \to -\infty$. The orbits enter the origin tangent to one linear orbit; they come from a direction parallel to the other linear orbit. We have labeled the linear orbits by their eigenvalues. The origin is an asymptotically stable node.

Example 4.34

(**Both eigenvalues positive**) In this case the exponential factors $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ go to infinity as $t \to +\infty$. Thus, both pairs of linear orbits approach infinity as $t \to +\infty$ along the rays defined by the two eigenvalues. From the general form of the solution, equation (4.34), *all* of the orbits of the linear system exit the origin at $t = -\infty$ and approach infinity as $t \to +\infty$. The phase plane diagram is very similar to the case of negative eigenvalues, but with the direction arrows reversed; they all go outward. Positive eigenvalues give a nodal structure and the origin is called an **unstable node**. Using a simple example we show that all the orbits leave the origin at $t = -\infty$ along the direction of the eigevector associated with the smallest positive eigenvalue. Suppose the solution of a system in component form is

$$x(t) = c_1 e^t + c_2 e^{4t},$$
$$y(t) = 2c_1 e^t - 2c_2 e^{4t},$$

where c_1 and c_2 are arbitrary constants. Then the eigenpairs are 1, $(1,2)^{\mathrm{T}}$ and 4, $(1,-2)^{\mathrm{T}}$. As $t \to -\infty$ (backward in time) we have $x(t) \to c_1 e^t$ and $y(t) \to 2c_1 e^t$ because e^t dominates e^{6t} when t is large and negative. Thus $y(t)/x(t) \to 2$ as $t \to -\infty$. Hence the curves at the origin are tangent to the direction $(1,2)$ of the eigenvector associated with the smallest eigenvalue near the origin. One can observe that all orbits approach the direction of the eigenvector associated with largest eigenvalue as $t \to +\infty$. □

The case $\det A = 0$

We have not discussed the case when zero is an eigenvalue, which is true if, and only if, $\det A = 0$. In this case all the critical points are non-isolated and lie on line $ax + by = 0$ (or $cx + dy = 0$, which is the same). This means the two equations are proportional and each point on the line is an equilibrium where the orbit where the solution remains for all time. Figure 4.10 shows the line of non-isolated critical points. Away from the line of equilibria, the two differential equations are proportional we can divide them to obtain

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by} = m,$$

where m is the constant of proportionality. Thus the orbits are $y = mx + C$, where C is a constant of integration. These are shown in Figure 4.10 as well. These orbits either all approach the line of equilibria or exit the line of equilibrium. To determine which is the case, we can calculate the sign of $x' = ax+by$ above and below the equilibrium line to determine if x is increasing or decreasing. A similar calculation could be made for the sign of y'. In a different way, if the nonzero eigenvalue is negative, then all the orbits approach the line of equilibria, and if it is positive, they exit the equilibria.

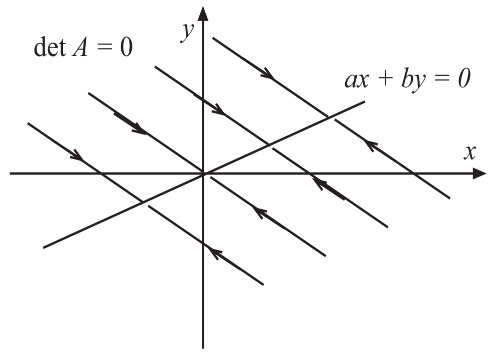

Figure 4.10 Phase plane diagram in the case $\det A = 0$ and the differential equations are proportional with $dy/dx = m$. The critical points are along the line $ax + by = 0$, and the other orbits are straight lines of slope m. *Either* all of the orbits approach the line of equilibria as $t \to +\infty$ (shown), *or* all of them exit the line of equilibria and go to infinity as $t \to +\infty$. In this latter case the arrows are reversed from those shown.

Example 4.35

Consider the system

$$x' = x - 2y,$$
$$y' = -2x + 4y.$$

Clearly the equations are proportional and $\det A = 0$. The equilibria are along the line $y = \frac{1}{2}x$. The orbits are found by dividing,

$$\frac{dy}{dx} = \frac{-2x + 4y}{x - 2y} = -2, \quad \text{or} \quad y = -2x.$$

The characteristic polynomial is $\lambda^2 - 5\lambda = 0$, giving eigenvalues $\lambda = 0, 5$. Therefore the orbits exit the the line of critical points. The phase diagram is similar to that in Figure 4.10 with the direction arrows reversed. □

Example 4.36

(**General solution**) Consider the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \mathbf{x}. \tag{4.37}$$

Notice that the coefficient matrix has $\det A = 0$ and so the characteristic equation is $\lambda^2 - 5\lambda = \lambda(\lambda - 5) = 0$, giving a zero eigenvalue, $\lambda_1 = 0$. This always occurs when $\det A = 0$. The other eigenvalue is $\lambda_2 = 5$. When $\lambda = 0$ then the eigenvectors are found by solving $(A - \lambda I)\mathbf{v} = A\mathbf{v} = \mathbf{0}$, or

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Again the equations are the same. Taking $v_1 + 2v_2 = 0$ gives $v_1 = -2v_2$; taking $v_2 = 1$ gives $v_1 = -2$. Therefore an eigenpair is

$$\lambda_1 = 0, \ \mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

The other eigenpair, left for the reader, is

$$\lambda_2 = 5, \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

A fundamental set of solutions is therefore

$$\mathbf{x}_1(t) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t},$$

and the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}.$$

The line of nonisolated critical points is the line in the direction of the eigenvector associated with $\lambda = 0$, or $x + 2y = 0$. There is one linear orbit $\mathbf{x}_2(t)$ leading to a pair of opposing rays that exit the origin along the line $y = \frac{1}{2}x$. All other orbits are parallel to the linear orbit with the same directions. □

EXERCISES

1. Using the following eigenpairs for a given system, write down the general solution and state the type of critical point and its stability.

   a) $-1, \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \ -2, \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

   b) $-1, \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \ 2, \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

   c) $1, \begin{pmatrix} -1 \\ 2 \end{pmatrix}; \ 2, \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

   d) $-3, \begin{pmatrix} 2 \\ 1 \end{pmatrix}; \ 2, \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$

2. Find the general solution of the systems with with the following coefficient matrices:

a) $\begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix}$. c) $\begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$.

b) $\begin{pmatrix} 1 & 0 \\ 3 & -4 \end{pmatrix}$. d) $\begin{pmatrix} 1 & 1 \\ -3 & 3 \end{pmatrix}$.

3. Sketch a phase plane diagram for the systems with coefficient matrices given in Exercise 2.

4. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

5. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

4.4.2 Complex Eigenvalues

If the eigenvalues of the matrix A are complex, there are solutions of the form

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}.$$

The eigenvalues, which are solutions of a quadratic equation, are complex conjugates, $\lambda = a \pm bi$. The corresponding eigenvectors are $\mathbf{v} = \mathbf{w} \pm i\mathbf{z}$, also complex conjugates. Taking the eigenpair $a+bi$, $\mathbf{w}+i\mathbf{z}$, we obtain the complex solution

$$(\mathbf{w} + i\mathbf{z})e^{(a+bi)t}.$$

Recalling that the real and imaginary parts of a complex solution are real solutions, we expand this complex solution using Euler's formula to get

$$\begin{aligned}(\mathbf{w} + i\mathbf{z})e^{at}e^{ibt} &= e^{at}(\mathbf{w} + i\mathbf{z})(\cos bt + i\sin bt) \\ &= e^{at}(\mathbf{w}\cos bt - \mathbf{z}\sin bt) + ie^{at}(\mathbf{w}\sin bt + \mathbf{z}\cos bt).\end{aligned}$$

Therefore two *real*, independent solutions are

$$\mathbf{x}_1(t) = e^{at}(\mathbf{w}\cos bt - \mathbf{z}\sin bt), \quad \mathbf{x}_2(t) = e^{at}(\mathbf{w}\sin bt + \mathbf{z}\cos bt),$$

and the **general solution** is a combination of these,

$$\mathbf{x}(t) = c_1e^{at}(\mathbf{w}\cos bt - \mathbf{z}\sin bt) + c_2e^{at}(\mathbf{w}\sin bt + \mathbf{z}\cos bt). \tag{4.38}$$

In the case of complex eigenvalues we need not consider both eigenpairs; each eigenpair leads to the same two independent solutions. For complex eigenvalues there are clearly no linear orbits, as you can see from the presence of the oscillatory trigonometric terms. The terms are periodic functions with frequency b and period $2\pi/b$, and they define orbits that rotate around the origin. The factor e^{at} acts as an amplitude factor and causes the rotating orbits to expand if $a > 0$, and thus we obtain expanding spiral orbits going away from the origin. If $a < 0$ the amplitude decays and orbits spiral into the origin.

In this case, we say the origin is a **spiral point**. If $a < 0$, the origin is an an **asymptotically stable spiral point**, and it is an attractor. When $a > 0$ the origin is a repeller and we say it is an **unstable spiral point.**

Example 4.37

(**Complex eigenvalues**) Let

$$\mathbf{x}' = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \mathbf{x}.$$

We calculated the eigenvalues and eigenvectors of the coefficient matrix in Example 4.28. The eigenvalues are $\lambda = -2 \pm 3i$ with corresponding eigenvectors $(-1, i)^{\mathrm{T}}$ and $(-1, -i)^{\mathrm{T}}$. Using $\lambda = -2 + 3i$ with eigenvector

$$\begin{pmatrix} -1 \\ i \end{pmatrix},$$

we get a complex solution. But, using Euler's formula we can obtain two linearly independent real solutions:

$$\begin{aligned} \mathbf{x}(t) &= \begin{pmatrix} -1 \\ i \end{pmatrix} e^{(-2+3i)t} = \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] e^{-2t}(\cos 3t + i \sin 3t) \\ &= \begin{pmatrix} -e^{-2t}\cos 3t \\ -e^{-2t}\sin 3t \end{pmatrix} + i \begin{pmatrix} -e^{-2t}\sin 3t \\ -e^{-2t}\cos 3t \end{pmatrix}. \end{aligned}$$

Therefore two linearly independent solutions are

$$\mathbf{x}_1(t) = \begin{pmatrix} -e^{-2t}\cos 3t \\ -e^{-2t}\sin 3t \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} -e^{-2t}\sin 3t \\ -e^{-2t}\cos 3t \end{pmatrix}.$$

The other complex conjugate eigenpair gives rise to the same two real solutions. So, the general solution of the system is a linear combination of these two solutions, $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$. In the phase plane the orbits are spirals that approach the origin as $t \to +\infty$ because the real part of the eigenvalues, -2, is negative. See Figure 4.11. At the point $(1, 1)$ the tangent vector is $(x', y') = (-5, 1)$, so the spirals turn in a counterclockwise manner. In this case the origin an **asymptotically stable spiral point**. □

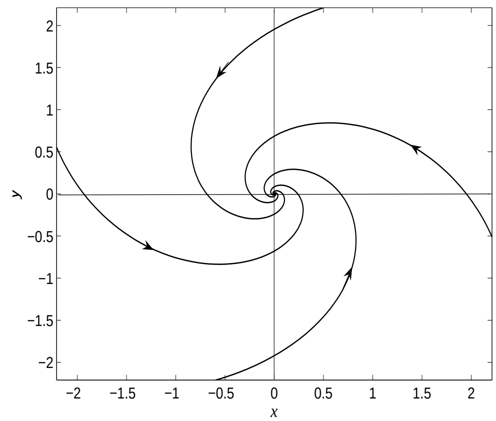

Figure 4.11 An asymptotically stable spiral from Example 4.37.

Purely Imaginary Eigenvalues

If the eigenvalues of A are **purely imaginary**, $\lambda = \pm\, bi$, then the amplitude factor e^{at} in (4.38) is absent and the solutions are periodic of period $2\pi/b$. The **general solution** is

$$\mathbf{x}(t) = c_1(\mathbf{w}\cos bt - \mathbf{z}\sin bt) + c_2(\mathbf{w}\sin bt + \mathbf{z}\cos bt).$$

The orbits are closed, periodic cycles and plot as concentric ellipses or circles. In this case we say the origin is a phase a **stable center**. Orbits close to the origin remain close, but no orbit approaches the origin.

EXERCISES

1. Write down the general solution of the linear system with the given eigenvalues and eigenvectors and express each solution in component form.

   a) $\lambda = \pm\, i,\;\; \mathbf{v} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$. b) $\lambda = 1 \pm\, 2i,\;\; \mathbf{v} = \begin{pmatrix} 1 \\ 1 \pm i \end{pmatrix}$.

2. Sketch a phase plane diagram for the system with coefficient matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}.$$

Include the nullclines (dashed) and sample direction field vectors.

3. Sketch a phase plane diagram for the system with coefficient matrix

$$\begin{pmatrix} 3 & 4 \\ 1 & -3 \end{pmatrix}.$$

Include the nullclines (dashed) and sample direction field vectors.

4.4.3 Real, Equal Eigenvalues

Next suppose the eigenvalues of the coefficient matrix A are real and equal, that is,

$$\lambda_1 = \lambda_2 \equiv \lambda,$$

and $\lambda \neq 0$. There are two cases.

In the case that λ has a pair of independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then there are two independent solutions

$$\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda t}, \qquad \mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda t}.$$

Therefore the general solutions is

$$\mathbf{x}_(t) = c_1 \mathbf{v}_1 e^{\lambda t} + c_2 \mathbf{v}_2 e^{\lambda t}.$$

This is a linear combination of two independent linear orbits.

In the second case, A has a repeated real eigenvalue λ with only a *single* eigenvector $\mathbf{v}$. In this case we say the matrix is **deficient**. Then

$$\mathbf{x}_1(t) = \mathbf{v} e^{\lambda t}$$

is one solution representing opposing linear orbits. However, we need another independent solution. An intuitive guess, based on experience with second-order equations in Chapter 2, is to take $te^{\lambda t}\mathbf{v}$, but that does not work. (Try it.) However, the slightly modified vector function

$$\mathbf{x}_2(t) = e^{\lambda t}(t\mathbf{v} + \mathbf{w}),$$

does satisfy the differential equation for a some constant vector $\mathbf{w}$ to be determined. Substituting $\mathbf{x}_2$ into the system we get

$$\begin{aligned} \mathbf{x}_2' &= e^{\lambda t}\mathbf{v} + \lambda e^{\lambda t}(t\mathbf{v} + \mathbf{w}), \\ A\mathbf{x}_2 &= e^{\lambda t} A(t\mathbf{v} + \mathbf{w}). \end{aligned}$$

These are equal and so we obtain an algebraic system for $\mathbf{w}$:

$$(A - \lambda I)\mathbf{w} = \mathbf{v}. \tag{4.39}$$

This system always has infinitely many solutions $\mathbf{w}$ because $\det(A - \lambda I) = 0$. We only need to choose one. The vector $\mathbf{w}$ is called a **generalized eigenvector** of A. Therefore we have determined a second linearly independent solution. In summary, the general solution to the linear system $\mathbf{x}' = A\mathbf{x}$ in the deficient case is

$$\mathbf{x}(t) = c_1\mathbf{v}e^{\lambda t} + c_2 e^{\lambda t}(t\mathbf{v} + \mathbf{w}).$$

where $\mathbf{w}$ is a solution to 4.39.

If the repeated eigenvalue is negative all the orbits enter the origin as $t \to +\infty$, and they go to infinity as $t \to -\infty$. We say the origin is an **asymptotically stable node**. If the eigenvalue is positive, the orbits reverse direction in time and all leave the origin; we say it is an **unstable node**. In the case of equal eigenvalues, the term **degenerate node** is often used to describe the structure, which appears different from a common node where the eigenvalues are unequal.

Example 4.38

Consider the system

$$\mathbf{x}' = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\mathbf{x}.$$

The characteristic polynomial is $\lambda^2 + 4\lambda + 4 = 0$, which gives equal eigenvalues $\lambda = -2, -2$. Then, to find the eigenvectors we substitute $\lambda = 2$ into the eigenvector equation

$$(A - \lambda I)\mathbf{v} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

There is no restriction on v_1 and v_2, other than both cannot be zero simultaneously; thus we can choose them arbitrarily. Take $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (0, 1)$, which are two independent eigenvectors associated with $\lambda = -2$. This is the non-deficient case. Two independent solutions are

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t}, \qquad \mathbf{x}_2(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}.$$

The general solution is

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}.$$

The components of the general solution are

$$x(t) = c_1 e^{-2t}, \;\; y(t) = c_2 e^{-2t}.$$

Taking their ratios we get

$$\frac{y(t)}{x(t)} = \frac{c_2 e^{-2t}}{c_1 e^{-2t}} = \frac{c_2}{c_1},$$

which gives the straight lines $y = \frac{c_2}{c_1}x$, for all c_1, c_2. These are the orbits for the system, *all* of which are rays, entering the origin as $t \to +\infty$. Note that $c_1 = 0$ corresponds to the two opposing vertical orbits on the y axis. These linear orbits can also be found by dividing the two original differential equations. □

Example 4.39

(**Deficient matrix**) Consider the system

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \mathbf{x}.$$

The eigenvalues are $\lambda = 3, 3$ and a corresponding eigenvector is $\mathbf{v} = (1, 1)^{\mathrm{T}}$. Therefore one solution is

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

Notice that this solution plots as a linear orbit coming out of the origin and approaching infinity along the direction $(1, 1)^{\mathrm{T}}$. There is automatically an opposite orbit coming out of the origin and approaching infinity along the direction $-(1, 1)^{\mathrm{T}}$. A second independent solution has the form $\mathbf{x}_2 = e^{3t}(t\mathbf{v} + \mathbf{w})$ where $\mathbf{w}$ satisfies

$$(A - 3I)\mathbf{w} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This equation has many solutions, and so we choose

$$\mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore a second solution has the form

$$\mathbf{x}_2(t) = e^{3t}(t\mathbf{v} + \mathbf{w}) = e^{3t} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} te^{3t} \\ (t+1)e^{3t} \end{pmatrix}.$$

The general solution of the system is the linear combination

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t).$$

If we append an initial condition, for example,

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then we can determine the two constants c_1 and c_2. We have

$$\mathbf{x}(0) = c_1\mathbf{x}_1(0) + c_2\mathbf{x}_2(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence

$$c_1 = 1, \quad c_2 = -1.$$

Therefore the solution to the initial value problem is given by

$$\mathbf{x}(t) = (1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + (-1) \begin{pmatrix} te^{3t} \\ (t+1)e^{3t} \end{pmatrix} = \begin{pmatrix} (1-t)e^{3t} \\ -te^{3t} \end{pmatrix}.$$

As time goes forward ($t \to +\infty$), the orbits go to infinity, and as time goes backward ($t \to -\infty$), the orbits enter the origin. The origin is an unstable node. □

EXERCISES

1. Find the general solution and sketch a phase diagram.

a) $\mathbf{x}' = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix} \mathbf{x}.$ b) $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}.$

2. Plot the solution curve found in Example 4.39 for $-2 \le t \le 1$.

4.5 Phase Plane Analysis

We developed all the tools to completely analyze two-dimensional linear systems, both analytically and graphically. In this section we take a broad overview of the process and apply it to some practical applications that contain parameters, not merely fixed numerical constants. A complete summary of the preceding sections is included in the discussion.

The analysis has shown that all of the analytic information about systems can be obtained by knowing the eigenvalues and eigenvectors of the coefficient matrix. The eigenvalues are roots of the characteristic equation

$$\lambda^2 - (\mathrm{tr}\, A)\lambda + \det A = 0$$

and are given by

$$\lambda = \frac{1}{2}\left(\mathrm{tr}\, A \pm \sqrt{(\mathrm{tr}\, A)^2 - 4 \det A}\right).$$

These depend only on knowledge of the trace and determinant of the matrix A. The following table summarizes the orbital structure and its dependence on the eigenvalues.

Table 4.1 Solution properties of the system $\mathbf{x}' = A\mathbf{x}$ in terms of the eigenvalues in the case $\det A \neq 0$.

Eigenvalues	***Orbital Structure***
λ_1, $\lambda_2 > 0$, real, unequal	unstable node
λ_1, $\lambda_2 < 0$, real, unequal	asymptotically stable node
$\lambda_1 < 0 < \lambda_2$, opposite signs	unstable saddle
$\lambda = \pm bi$, purely imaginary	center; stable ellipses or circles
$\lambda = a \pm bi$, $a > 0$	unstable spiral
$\lambda = a \pm bi$, $a < 0$	asymptotically stable spiral
$\lambda_1 = \lambda_2 > 0$	unstable node
$\lambda_1 = \lambda_2 < 0$	asymptotically stable node

The basic types of geometrical structures in the phase plane are shown in Figure 4.12.

The following theorem summarizes the dependence on the trace and determinant.

Theorem 4.40

Consider the linear system $\mathbf{x}' = A\mathbf{x}$, where $\det A \neq 0$. Then

- If $\det A < 0$, then $(0, 0)$ is a saddle point.
- If $\det A > 0$ and $\operatorname{tr} A = 0$, then $(0, 0)$ is a center.
- If $\det A > 0$, $\operatorname{tr} A > 0$, and $4 \det A \leq \operatorname{tr} A$, then $(0, 0)$ is an unstable node.
- If $\det A > 0$ $\operatorname{tr} A > 0$, and $4 \det A > \operatorname{tr} A$, then $(0, 0)$ is an unstable spiral.
- If $\det A > 0$, $\operatorname{tr} A < 0$, and $4 \det A \leq \operatorname{tr} A$, then $(0, 0)$ is an asymptotically stable node.
- If $\det A > 0$ $\operatorname{tr} A < 0$, and $4 \det A > \operatorname{tr} A$, then $(0, 0)$ is an asymptotically unstable spiral. □

Figure 4.13 shows the conclusions of the theorem on a plot of $\det A$ versus $\operatorname{tr} A$. One can compute these two quantities and locate the point on the plot to quickly determine the type of structure and its stability.

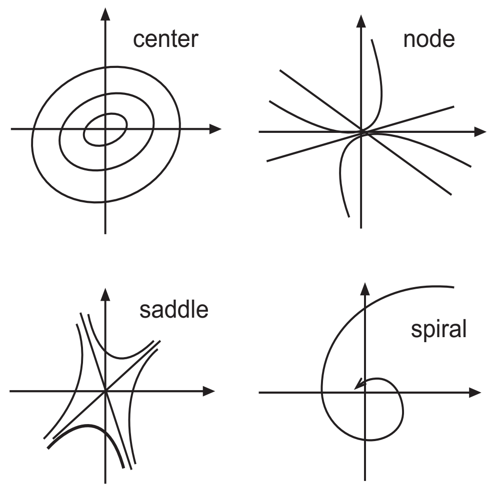

Figure 4.12 Generic plots of the four basic orbital structures for two-dimensional linear systems: center, node, saddle, and spiral, in the case det $A \neq 0$. Orbital directions are not indicated. If the eigenvalues are equal, the node has a different appearance from that shown.

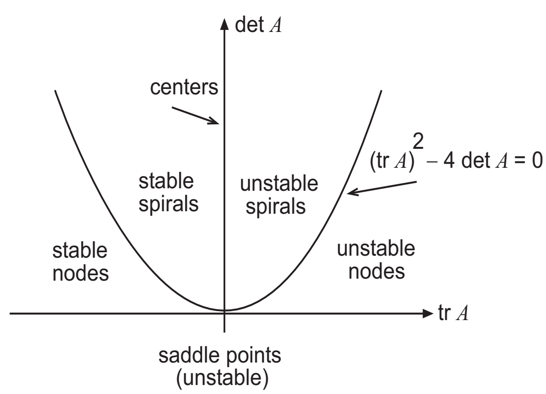

Figure 4.13 A plot of the regions of the *trace–determinant plane* where the various orbital structures occur. Along the horizontal axis where det $A = 0$, one of the eigenvalues is zero, giving an exceptional case of a line of nonisolated equilibria.

Finally, we summarize our results with the stability theorem for two-dimensional linear systems $\mathbf{x}' = A\mathbf{x}$.

Theorem 4.41

The origin is asymptotically stable if, and only if,

$$\operatorname{tr} A < 0 \quad \text{and} \quad \det A > 0. \quad \square$$

How to Draw a Phase Diagram

To draw a rough phase diagram for the linear system $\mathbf{x}' = A\mathbf{x}$, all we need to know are the eigenvalues, the eigenvectors, the direction field, and the nullclines where the vector field is vertical, or horizontal.

(a) If the eigenvalues are real and unequal, draw the linear orbits which are in the direction of the associated eigenvectors. Label each ray, or linear orbit, with an arrow that points toward the origin if the eigenvalue is negative and away from the origin if the eigenvalue is positive. Using the direction field, fill in the regions between these linear orbits with consistent solution curves, paying attention to which "eigen-direction" dominates as $t \to \infty$ and $t \to -\infty$ Drawing nullclines refines the orbital plots.

(b) If the eigenvalues are purely imaginary then the orbits are closed ellipses around the origin. Compute a direction field vector to determine if the orbits are clockwise or counterclockwise. Draw in the nullclines to obtain the fine detail where the closed orbits are vertical or horizontal.

(c) If the eigenvalues are complex then the orbits are spirals. If the real part of the eigenvalues is negative, they spiral in; if the real part is positive, they spiral out. Compute a direction field vector to determine if the spirals are clockwise or counterclockwise and draw in the nullclines to obtain the finer detail.

(d) If zero is an eigenvalue ($\det A = 0$) there is a line of equilibria or critical points along $ax + by = 0$. Divide the differential equations and solve to find the orbits, which are straight lines approaching the equilibria or exiting the equilibria.

The next four examples illustrate classical applications of linear systems and how to analyze them geometrically.

Example 4.42

(Crop-Soil Model) In this example we return to the crop-soil compartmental model introduced in Section 4.1 (see Example 4.2). Figure 4.14 indicates the flow rates between the two compartments. If $x = x(t)$ is the amount of

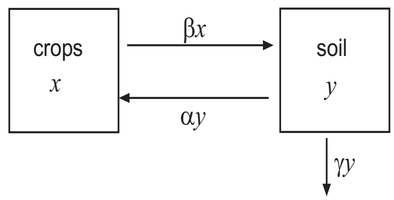

Figure 4.14 A compartmental diagram showing the exchange rates of the herbicide between crops and the soil. The $-\gamma y$ term represents the degradation, or decay, rate in the soil.

pesticide in the crop, and $y = y(t)$ is the amount in the soil, we obtained a two-dimensional system with unknowns $x(t)$ and $y(t)$:

$$\frac{dx}{dt} = -\beta x + \alpha y,$$
$$\frac{dy}{dx} = \beta x - (\alpha + \gamma)y.$$

The coefficient matrix is

$$A = \begin{pmatrix} -\beta & \alpha \\ \beta & -(\alpha+\gamma) \end{pmatrix}$$

Dividing these equations leads to a difficult differential equation or x and y, which we want to avoid. Also, finding the general solution in terms of eigenvalues and eigenvectors of A involves many cases because all we know is that the parameters are positive. Therefore, a geometric, or qualitative, approach is a good strategy to find the behavior of the system.

Setting both equations equal to zero shows that the only critical point is the origin, $x = y = 0$. Because x and y are nonnegative, only the first quadrant is relevant. To get an indication of the direction field we plot the x *nullcline* where $x' = 0$, or $-\beta x + \alpha y = 0$. The vector field is vertical this line so orbits must cross vertically. Next we plot the y *nullcline*) where $y' = 0$ or $\beta x - (\alpha+\gamma)y = 0$. The vector field is horizontal along that line, so orbits cross horizontally. Therefore, the x and y nullclines are the straight lines

$$y = \frac{\beta}{\alpha}x \quad (x\text{-nullcline}); \qquad y = \frac{\beta}{\alpha+\gamma}x \quad (y\text{-nullcline})$$

The nullclines are dashed lines in Figure 4.15. The x-nullcline has greater slope than the y-nullcline. In the three regions bounded by the nullclines we can calculate the direction of the vector field. For example, $x' = \alpha y - \beta x > 0$ whenever $y > (\beta/\alpha)x$, and $x' = \alpha y - \beta x < 0$ whenever $y < (\beta/\alpha)x$. Similarly, $y' = \beta x - (\alpha+\gamma)y > 0$ whenever $y < (\beta/(\alpha+\gamma))x$, and $y' = \beta x - (\alpha+\gamma)y < 0$ whenever $y < (\beta/(\alpha+\gamma))x$. Putting this information together in a single plot, we see that the vector field points *southeast* in the top most region, *southwest* in the region between the nullclines, and *northwest* in the lower region.

Figure 4.16 (right panel) shows the behavior of the orbits as determined by the vector field. It is clear that the origin is an *asymptotically stable node*. This means the pesticide eventually disappears from the system. A corresponding time series plot for the lower orbit in the left panel confirms that the pesticide concentrations in both the plants and the soil eventually decay to zero.

To confirm these results analytically, we compute $\operatorname{tr} A = -(\alpha+\beta+\gamma) < 0$ and $\det A = \beta(\alpha+\gamma) - \alpha\beta = \beta\gamma > 0$. Therefore, by Theorem 4.41, the origin is asymptotically stable. Therefore it is a node or spiral point. By our diagram it is a node because the vector field has no rotation.

In summary, we analyzed this problem geometrically using the direction field without solving the equations at all. Often this is all we want from a model. □

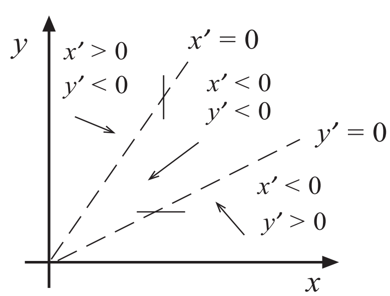

Figure 4.15 The x-nullcline ($x' = 0$) and the y-nullcline ($y' = 0$) shown dashed; note that the y-nullcline is below the x-nullcline. In the three regions bounded by the nullclines the direction field is indicated.

Example 4.43

We give an example of sketching the phase diagram using chiefly geometric methods. Consider

$$x' = -7x + 6y,$$
$$y' = 6x + 2y.$$

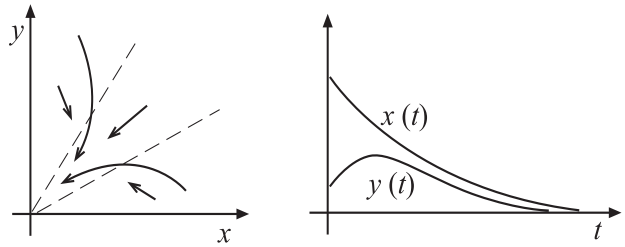

Figure 4.16 (Left) The orbits, as determined by the direction field in Figure 4.15. (Right) A sample time series plot corresponding to the lower orbit (right) in Figure 4.15.

Immediately we see $\det A < 0$ and therefore $(0,0)$ is a saddle point. To get the fine detail we plot the nullclines. We have $x' = 0$ on $y = -(7/6)x$ (x-nullcline), and $y' = 0$ on $y = -3x$ (y-nullcline). See Figure 4.17. There are 4 regions bounded by the nullclines. In each region we choose a point and find the direction field in the entire region. For example, take the point $(0,1)$ on the positive y axis; the direction field is $(x', y') = (6, 2)$, which is *northeast*, and it has a NE direction in the entire region. The direction field in the other 3 regions is shown in the figure. Using knowledge that $(0,0)$ is a saddle point, we can place four key orbits on the phase diagram to indicate the flow. We know there are 4 rays, or separatrices, that separate the types of orbits. We cannot know the exact directions of these rays unless we compute the eigenvectors. The separatrices entering the origin are associated with the negative eigenvalue, and the separatrices exiting the origin are associated with the positive eigenvalue. □

Example 4.44

(**Glucose–Insulin Interaction**) When an individual consumes food, especially carbohydrates, the pancreas responds by producing insulin, the key hormone that unlocks cell receptors on the cell walls to inject the glucose into the cell. If x denotes the excess amount of glucose in the blood above some equilibrium amount, and y denotes the excess amount of insulin, then a very simplified model of the dynamics of the interaction is

$$x' = -gx - ry,$$
$$y' = sx - dy,$$

where g is the natural decay rate of glucose (e.g., in excretion), r is the rate that insulin decreases the glucose, s is the rate that insulin production is stimulated

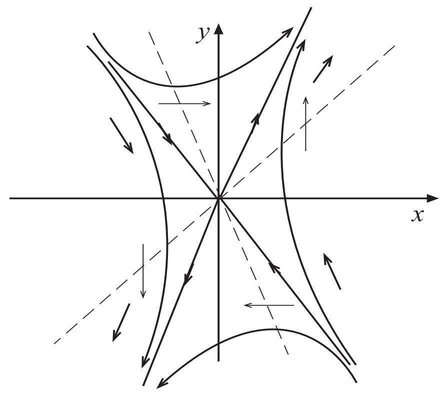

Figure 4.17 The x-nullclines ($x' = 0$) and the y-nullcline ($y' = 0$) shown dashed. In the 4 regions bounded by the nullclines the direction field is indicated. Approximate separatrices are drawn in to in to indicate the linear orbits or separatrices.

by the presence of glucose, and d is the natural decay rate of insulin. The constants are positive. (Note that x or y may be negative if the amounts are below equilibrium values.) Representative experimental values are

$$g = 2.9,\ r = 4.3,\ s = 0.21,\ d = 0.78. \tag{4.40}$$

The origin is the only equilibrium because $\det A = (-g)(-d) - s(-r) > 0$. The x nullcline, $y = -(g/r)x$, where the direction field is vertical, has negative slope, and the y nullcline, $y = (s/d)x$, where the direction field is horizontal, has positive slope. See Figure 4.18. In the 4 regions bounded by the nullclines the direction field is easily determined and shown in the figure. For example, $x' > 0$ if $y < -(g/r)x$, or underneath the x nullcline; and $x' < 0$ if $y > -(g/r)x$, or above the x nullcline. From these simple ideas we can get a rough idea of the shapes of the orbits. There appears to be a counterclockwise rotation, which may mean a spiral or center; or, the orbits could just veer into the origin, making it a stable node. We must work harder to determine which is true.

The characteristic equation is

$$\lambda^2 - (\mathrm{tr}\, A)\lambda + \det A = \lambda^2 + (g+d)\lambda + (rs + dg) = 0,$$

which gives the eigenvalues

$$\begin{aligned} \lambda &= \frac{1}{2}\left(-(g+d) \pm \sqrt{(g+d)^2 - 4(rs+dg)}\,\right) \\ &= \frac{1}{2}\left(-(g+d) \pm \sqrt{(g-d)^2 - 4rs}\right). \end{aligned}$$

Different cases can occur for different values of the parameters. First, notice that $\operatorname{tr} A < 0$ and $\det A > 0$, so the origin is asymptotically stable. Therefore, there can be no purely oscillatory solutions where the origin is a center. If $(g-d)^2 < 4rs$, the roots are complex with negative real part, and we obtain decaying oscillations, or a stable spiral. If $(g-d)^2 > 4rs$, then the roots are real and negative, giving a stable node. Stable spirals occur when rs is large, meaning

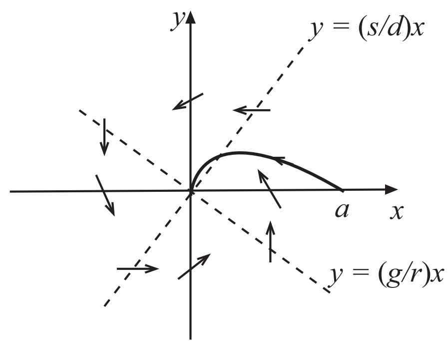

Figure 4.18 Glucose–insulin dynamics. The nullclines and direction field are shown. A typical orbit beginning at $(x, y) = (a, 0)$ at time $t = 0$ is shown for the case of a node, which occurs when $(g-d)^2 > 4rs$. The system admits stable spirals when $(g+d)^2 < 4rs$.

there is an excessive insulin response to the glucose, causing a hypoglycemic individual's excess glucose to overshoot and become negative. The reader is asked in an exercise to determine the solution for the parameters given in (4.40). □

Example 4.45

(**Circuits**) Electrical circuits provide important examples of linear systems. Figure 4.19 shows a circuit with two loops, and the current in the three sections is shown. Their directions can be chosen arbitrarily as long as we are consistent in setting up the circuit equations. At any node (say **n**) the sum of the currents entering the node must equal the sum of the currents leaving the node. In this case,

$$I_1 + I_2 = I_3.$$

Notice the same equation holds at the opposite node. Now we apply Kirchhoff's law in each loop. There are no voltage sources, or emfs. If we traverse the top

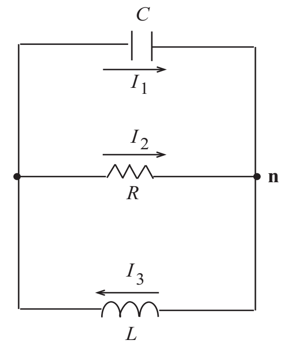

Figure 4.19 A two-loop circuit with a capacitor, resistor, and inductor. The direction of currents is shown.

loop in a clockwise direction, then the sum of the voltage drops across each element is zero, or

$$\frac{1}{C}Q - RI_2 = 0,$$

where Q is the charge on the capacitor, and $Q' = I_1$. The negative sign occurs because we are traversing the resistor in the opposite direction of the given current. In the lower loop, again clockwise, Kirchhoff's law gives

$$RI_2 + LI_3' = 0.$$

It always takes some effort to reduce the equations to a linear system. Here we can easily eliminate I_1, which is $I_1 = I_3 - I_2$. Differentiating the equation for the first loop gives

$$\frac{1}{C}I_1 - RI_2' = 0, \quad \text{or} \quad \frac{1}{C}(I_3 - I_2) - RI_2' = 0.$$

Solving the last two equations for the derivatives gives a linear system for I_2 and I_3,

$$I_2' = \frac{1}{RC}(-I_2 + I_3),$$
$$I_3' = -\frac{R}{L}I_2.$$

Therefore the coefficient matrix is

$$A = \begin{pmatrix} -\frac{1}{RC} & \frac{1}{RC} \\ -\frac{R}{L} & 0 \end{pmatrix}.$$

Easily, $\operatorname{tr} A = -\frac{1}{RC} < 0$ and $\det A = \frac{1}{LC} > 0$. Thus the origin is asymptotically stable. We note that $(\operatorname{tr} A)^2 - 4\det A = \frac{1}{(RC)^2} - \frac{4}{LC}$. Therefore:

$$\text{(a) if } \frac{L}{R^2C} > 4 \text{ then } (0,0) \text{ a node}$$
$$\text{(b) if } \frac{L}{R^2C} < 4 \text{ then } (0,0) \text{ a spiral}$$

Figure 4.20 shows a typical orbit in the case of a spiral point.

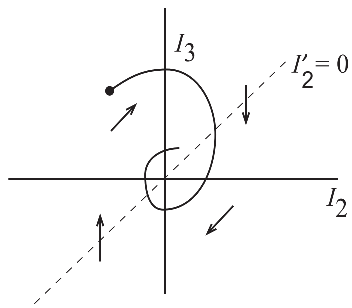

Figure 4.20 A typical decaying spiral orbit for the circuit shown in Figure 4.19 representing decaying oscillations.

Example 4.46

(**Chemical tank reactors**) Figure 4.21 shows two tanks with equal volumes V and equal inflow and outflow volumetric rates q, in volume per time. We let $C_1 = C_1(t)$ and $C_2 = C_2(t)$ denote the concentrations (mass per volume) of the chemical in each tank. Further, q_1 and q_2 denote the volumetric flow rate (volume per time) of the chemical from Tank 1 to Tank 2, and from Tank 2 to Tank 1, respectively. The concentration of the chemical entering Tank 1 is c_{in}. Note that a *volumetric flow rate times a concentration* has units of mass per time.

The conservation of mass law (see Chapter 1), which is applied to each reactor, is

$$\frac{d}{dt}(\text{Total mass in tank}) = \text{rate mass flows in} - \text{rate mass flows out}$$

Therefore, applying conservation of mass to each reactor we have

$$VC_1' = qc_{\text{in}} - q_1C_1 + q_2C_2,$$
$$VC_2' = q_1C_1 - q_2C_2 - qC_2,$$

which is a linear, nonhomogeneous system for C_1 and C_2. If $c_{\text{in}} = 0$, then the system is homogeneous and has the form

$$C_1' = -\frac{q_1}{V}C_1 + \frac{q_2}{V}C_2, \tag{4.41}$$

$$C_2' = \frac{q_1}{V}C_1 - \left(\frac{q_2 + q}{V}\right)C_2. \tag{4.42}$$

The coefficient matrix is

$$A = \begin{pmatrix} -\frac{q_1}{V} & \frac{q_2}{V} \\ \frac{q_1}{V} & -\frac{q_2+q}{V} \end{pmatrix}.$$

The trace is clearly negative and the determinant ($= q_1q/V^2$) is positive. Therefore the origin is asymptotically stable. Therefore the concentrations eventually go to zero. □

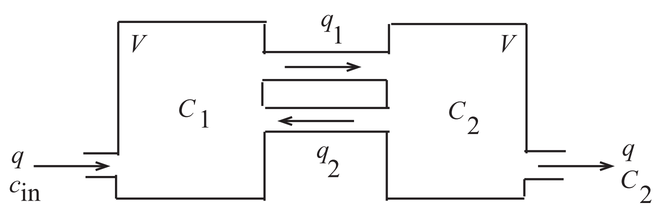

Figure 4.21 Coupled chemical reactors.

EXERCISES

1. Find the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}; \quad B = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}; \quad C = \begin{pmatrix} 2 & -8 \\ 1 & -2 \end{pmatrix}.$$

2. Write the general solution of the linear system $\mathbf{x}' = A\mathbf{x}$ if A has eigenpairs 2, $(1,5)^{\mathrm{T}}$ and -3, $(2,-4)^{\mathrm{T}}$. Sketch the linear orbits in the phase plane corresponding to these eigenpairs. Find the solution curve that satisfies the initial condition $\mathbf{x}(0) = (0,1)^{\mathrm{T}}$ and plot it in the phase plane. Do the same for the initial condition $\mathbf{x}(0) = (-6,12)^{\mathrm{T}}$.

3. Answer the questions in Exercise 2 for a system whose eigenpairs are -6, $(1,2)^{\mathrm{T}}$ and -1, $(1,-5)^{\mathrm{T}}$.

4. For each system find the general solution and sketch the phase portrait. Indicate the linear orbits (if any) and the direction of the solution curves.

   a) $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\mathbf{x}.$

   b) $\mathbf{x}' = \begin{pmatrix} -3 & 4 \\ 0 & -3 \end{pmatrix}\mathbf{x}.$

   c) $\mathbf{x}' = \begin{pmatrix} 2 & 2 \\ 6 & 3 \end{pmatrix}\mathbf{x}.$

   d) $\mathbf{x}' = \begin{pmatrix} -5 & 3 \\ 2 & -10 \end{pmatrix}\mathbf{x}.$

   e) $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x}.$

   f) $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}\mathbf{x}.$

   g) $\mathbf{x}' = \begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix}\mathbf{x}.$

   h) $\mathbf{x}' = \begin{pmatrix} 0 & 9 \\ -1 & 0 \end{pmatrix}\mathbf{x}.$

5. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

6. Consider the system

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\mathbf{x}.$$

   a) Find the equilibrium solutions and plot them in the phase plane.

   b) Find the eigenvalues and determine if there are linear orbits.

   c) Find the general solution and plot the phase portrait.

7. (**Bifurcation**) Bifurcation theory is the study of how equilibrium solutions and their stability properties change as parameters change in a problem. The the following system let α be a parameter:

$$\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} \mathbf{x}.$$

   a) Find the general solution when $\alpha = 1/2$, and classify the critical point.

   b) Repeat if $\alpha = 2$.

   c) Discuss the behavior of the system when $1/2 < \alpha < 2$.

8. For the following systems: (i) Determine the eigenvalues in terms of α. (ii) Find the values of α for which the phase plane changes type. (iii) Sketch a typical phase diagram.

   a) $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix} \mathbf{x}.$ b) $\mathbf{x}' = \begin{pmatrix} 4 & \alpha \\ 8 & -6 \end{pmatrix} \mathbf{x}.$

9. Determine the behavior of solutions near the origin for the system

$$\mathbf{x}' = \begin{pmatrix} 3 & a \\ 1 & 1 \end{pmatrix} \mathbf{x}$$

   for different values of the parameter a.

10. The matrix for a linear system is

$$\begin{pmatrix} a & \sqrt{2} + a/2 \\ \sqrt{2} - a/2 & 0 \end{pmatrix},$$

   where $-\infty < a < \infty$. Sketch the path traced out in the trace-determinant plane as a varies over its range. Compute the values of a where a bifurcation occurs and state what the bifurcation is.

11. For the systems in Exercise 4, characterize the origin as to type (node, center, spiral, saddle) and stability (unstable, neutrally stable, asymptotically stable).

12. Consider the system

$$x' = -3x + ay,$$
$$y' = bx - 2y.$$

   Are there values of a and b where the solutions are closed cycles (periodic orbits)?

13. In the tank reactor setup in Example 4.46, take $c_{\text{in}} = 0$, $q = 15$ liters/min, $q_1 = 16$ liters/min, $q_2 = 1$ liter/min, with $V = 4$ liters. If there is no salt in tank 1 and 0.3 oz in tank 2, find the concentrations and sketch a phase plane diagram.

14. For each of the following nonhomogeneous systems: **(a)** Find the equilibrium solutions, if any. **(b)** Find the general solution. **(c)** Sketch a phase diagram.

a) $\mathbf{x}' = \begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ c) $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 3 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

b) $\mathbf{x}' = \begin{pmatrix} 3 & 4 \\ 1 & -3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ d) $\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -3 & -3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

15. Consider the circuit shown in Figure 4.22 with two loops. I_1 and I_2 are the currents in the upper and lower loops and I_2 is the current in the middle segment, with directions as shown. $E(t)$ is the electromotive force.

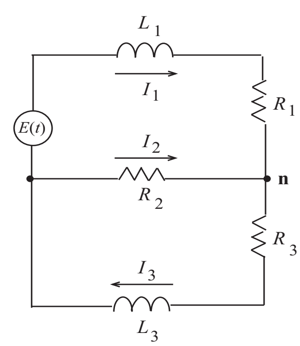

Figure 4.22 Two-loop circuit.

a) Apply Kirchhoff's law in both loops to obtain

$$L_1 I_1' + R_1 I_1 - R_2 I_2 = E(t), \quad L_3 I_3' + R_2 I_2 + R_3 I_3 = 0.$$

b) The sum of the currents entering the node, or junction, **n** must equal the sum of the currents leaving the node. Show

$$I_1 + I_2 = I_3.$$

c) Eliminate the current from the two equations in part (a) and derive a nonhomogeneous linear system for I_1, I_3. Write the system in standard form, identifying the coefficient matrix.

d) Sketch phases diagrams for each possibility. The diagrams should include critical points, nullclines, the direction field, and sample orbits.

16. Analyze the circuit in Figure 4.23, letting I be the current through the inductor and V the voltage across the capacitor. Begin by showing the governing equations are

$$I' = -I - V, \quad V' = 2I - V.$$

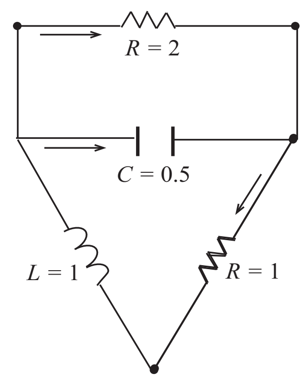

Figure 4.23 circuit.

17. A system of the form

$$\mathbf{x}'(t) = rA\mathbf{x}(t - T),$$

where the growth rate at time t depends on the state of the system at an earlier time $t - T$ is called a *delay equation*. Show that the system has a solution of the form

$$\mathbf{x}(t) = \mathbf{v}e^{\sigma t}$$

provided $\mathbf{v}$ is an eigenvector of A (with eigenvalue λ). In particular, show that σ must satisfy the equation

$$\sigma = r\lambda e^{-\sigma T}.$$

Will this equation always have a solution σ?

4.6 Nonhomogeneous Systems

Corresponding to a two-dimensional, linear homogeneous system $\mathbf{x}' = A\mathbf{x}$, we now examine the **nonhomogeneous system**

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t), \tag{4.43}$$

where

$$\mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$$

is a prescribed vector function. We think of the function $\mathbf{f}(t)$ as the driving force, or a source, in the system. In previous sections we examined problems with $\mathbf{f}(t) = \mathbf{b}$, where $\mathbf{b}$ is a constant force.

To ease the notation in writing the solution of (4.43) we define a **fundamental matrix** $\Phi(t)$ as a 2×2 matrix whose columns are two independent solutions to the associated homogeneous system $\mathbf{x}' = A\mathbf{x}$. So, the fundamental matrix is a square array that holds both vector solutions. It is straightforward to show that $\Phi(t)$ satisfies the *matrix* equation $\Phi'(t) = A\Phi(t)$, and that the general solution to the homogeneous equation $\mathbf{x}' = A\mathbf{x}$ can is form

$$\mathbf{x}_h(t) = \Phi(t)\mathbf{c},$$

where $\mathbf{c} = (c_1, c_2)^{\mathrm{T}}$ is an arbitrary constant vector. (The reader should do Exercise 1 presently, which requires verifying these relations.)

The variation of parameters method introduced in Chapter 2 for nonhomogeneous, second-order equations is applicable to first-order linear systems. Therefore we assume a solution to (4.43) of the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{c}(t), \tag{4.44}$$

where we have "varied" the constant vector $\mathbf{c}$. Then, using the product rule for differentiation (which works for matrices),

$$\begin{aligned} \mathbf{x}'(t) &= \Phi(t)\mathbf{c}'(t) + \Phi'(t)\mathbf{c}(t) = \Phi(t)\mathbf{c}'(t) + A\Phi(t)\mathbf{c}(t) \\ &= A\mathbf{x} + \mathbf{f}(t) = A\Phi(t)\mathbf{c}(t) + \mathbf{f}(t). \end{aligned}$$

Comparison gives

$$\Phi(t)\mathbf{c}'(t) = \mathbf{f}(t) \quad \text{or} \quad \mathbf{c}'(t) = \Phi(t)^{-1}\mathbf{f}(t).$$

We can invert the fundamental matrix because its determinant is nonzero, a fact that follows from the independence of its columns. Integrating the last equation from 0 to t then gives

$$\mathbf{c}(t) = \int_0^t \Phi(s)^{-1}\mathbf{f}(s)ds + \mathbf{k},$$

where $\mathbf{k}$ is a arbitrary constant vector. Note that the integral of a vector function is defined to be the vector consisting of the integrals of the components. Substituting into (4.44) shows that the general solution to the nonhomogeneous equation (4.43) is

$$\mathbf{x}(t) = \Phi(t)\mathbf{k} + \Phi(t)\int_0^t \Phi(s)^{-1}\mathbf{f}(s)ds. \tag{4.45}$$

As is the case for a single first-order linear DE, this formula gives the general solution of (4.43) as a sum of the general solution to the homogeneous equation (first term) and a particular solution to the nonhomogeneous equation (second term). Equation (4.45) is called the **variation of parameters formula** for systems. It is equally valid for systems of any dimension, with appropriate size increase in the vectors and matrices.

It is sometimes a formidable task to calculate the solution (4.45), even in the two-dimensional case. It involves finding the two independent solutions to the homogeneous equation, forming the fundamental matrix, inverting the fundamental matrix, and then integrating.

Example 4.47

Consider the nonhomogeneous system

$$\mathbf{x}' = \begin{pmatrix} 4 & 3 \\ -1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

It is a straightforward exercise to find the solution to the homogeneous system

$$\mathbf{x}' = \begin{pmatrix} 4 & 3 \\ -1 & 0 \end{pmatrix}\mathbf{x}.$$

The eigenpairs are 1, $(1,-1)^{\mathrm{T}}$ and 3, $(-3,1)^{\mathrm{T}}$. Therefore two independent solutions are

$$\begin{pmatrix} e^t \\ -e^t \end{pmatrix}, \begin{pmatrix} -3e^{3t} \\ e^{3t} \end{pmatrix}.$$

A fundamental matrix is

$$\Phi(t) = \begin{pmatrix} e^t & -3e^{3t} \\ -e^t & e^{3t} \end{pmatrix},$$

and its inverse is

$$\Phi^{-1}(t) = \frac{1}{\det\Phi}\begin{pmatrix} e^{3t} & 3e^{3t} \\ e^t & e^t \end{pmatrix} = \frac{1}{-2e^{4t}}\begin{pmatrix} e^{3t} & 3e^{3t} \\ e^t & e^t \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} e^{-t} & 3e^{-t} \\ e^{-3t} & e^{-3t} \end{pmatrix}.$$

By the variation of parameters formula (4.45), the general solution is

$$\begin{aligned}
\mathbf{x}(t) &= \Phi(t)\mathbf{k} + \begin{pmatrix} e^t & -3e^{3t} \\ -e^t & e^{3t} \end{pmatrix} \int_0^t -\frac{1}{2}\begin{pmatrix} e^{-s} & 3e^{-s} \\ e^{-3s} & e^{-3s} \end{pmatrix}\begin{pmatrix} 0 \\ s \end{pmatrix} ds \\
&= \Phi(t)\mathbf{k} - \frac{1}{2}\begin{pmatrix} e^t & -3e^{3t} \\ -e^t & e^{3t} \end{pmatrix} \int_0^t \begin{pmatrix} 3se^{-s} \\ se^{-3s} \end{pmatrix} ds \\
&= \Phi(t)\mathbf{k} - \frac{1}{2}\begin{pmatrix} e^t & -3e^{3t} \\ -e^t & e^{3t} \end{pmatrix}\begin{pmatrix} 3\int_0^t se^{-s}ds \\ \int_0^t se^{-3s}ds \end{pmatrix} \\
&= \Phi(t)\mathbf{k} - \frac{1}{2}\begin{pmatrix} e^t & -3e^{3t} \\ -e^t & e^{3t} \end{pmatrix}\begin{pmatrix} 3-3(t+1)e^{-t} \\ \frac{1}{9}-(\frac{t}{3}+\frac{1}{9})e^{-3t} \end{pmatrix} \\
&= \begin{pmatrix} k_1e^t - 3k_2e^{3t} \\ -k_1e^t + k_2e^{3t} \end{pmatrix} + \begin{pmatrix} t+\frac{4}{3} \\ -\frac{4}{3}t - \frac{13}{9} \end{pmatrix}. \qquad \square
\end{aligned}$$

If the nonhomogeneous term $\mathbf{f}(t)$ is relatively simple, we can use the method of *undetermined coefficients* (judicious guessing) introduced for second-order equations in Chapter 2 to find the particular solution. In this case we guess the trial form of a particular solution, depending upon the form of $\mathbf{f}(t)$. For example, if both components are polynomials, then we guess a particular solution with both components being polynomials that have the highest degree that appears.

Example 4.48

If

$$\mathbf{f}(t) = \begin{pmatrix} 1 \\ t^2+2 \end{pmatrix},$$

then a guess for the particular solution would be

$$\mathbf{x}_p(t) = \begin{pmatrix} a_1t^2 + b_1t + c_1 \\ a_2t^2 + b_2t + c_2 \end{pmatrix}.$$

Substitution into the nonhomogeneous system then determines the six constants. □

Generally, if a term appears in one component of $\mathbf{f}(t)$, then the guess must have that term appear in all its components. The method is successful on forcing terms with sines, cosines, polynomials, exponentials, and products and sums of those. The rules are the same as for single equations. But the calculations are tedious and a computer algebra system is often preferred.

Example 4.49

We use the method of undetermined coefficients to find a particular solution to the equation in Example 4.47. The forcing function is

$$\begin{pmatrix} 0 \\ t \end{pmatrix},$$

and therefore we guess a particular solution of the form

$$\mathbf{x}_p = \begin{pmatrix} at+b \\ ct+d \end{pmatrix}.$$

Substituting into the original system yields

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} at+b \\ ct+d \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

Simplifying leads to the two equations

$$\begin{aligned} a &= (4a+3c)t + 4b + 3d, \\ c &= -b + (1-a)t. \end{aligned}$$

Comparing coefficients gives

$$a = 1, \quad b = -c = \frac{4}{3}, \quad d = -\frac{13}{9}.$$

Therefore a particular solution is

$$\mathbf{x}_p = \begin{pmatrix} t + \frac{4}{3} \\ -\frac{4}{3}t - \frac{13}{9} \end{pmatrix}. \qquad \square$$

EXERCISES

1. Let

$$\mathbf{x}_1 = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}$$

be independent solutions to the homogeneous equation $\mathbf{x}' = A\mathbf{x}$, and let

$$\Phi(t) = \begin{pmatrix} \phi_1(t) & \psi_1(t) \\ \phi_2(t) & \psi_2(t) \end{pmatrix}$$

be a fundamental matrix. Show, by direct calculation and comparison of entries, that $\Phi'(t) = A\Phi(t)$. Show that the general solution of the homogeneous system can be written equivalently as

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \Phi(t)\mathbf{c},$$

where $\mathbf{c} = (c_1, c_2)^{\mathrm{T}}$ is an arbitrary constant vector.

2. Two lakes of volume V_1 and V_2 initially have no contamination. A toxic chemical flows into lake 1 at $q + r$ gallons per minute with a concentration c grams per gallon. From lake 1 the mixed solution flows into lake 2 at q gallons per minute, and it simultaneously flows out into a drainage ditch at r gallons per minute. In lake 2 the the chemical mixture flows out at q gallons per minute. **(a)** If x and y denote the concentrations of the chemical in lake 1 and lake 2, respectively, set up an initial value problem whose solution would give these two concentrations (draw a compartmental diagram). **(b)** What are the equilibrium concentrations in the lakes, if any? Find $x(t)$ and $y(t)$. **(c)** Now change the problem by assuming the initial concentration in lake 1 is x_0 and fresh water flows in. Write down the initial value problem and qualitatively, without solving, describe the dynamics of this problem using eigenvalues.

3. Solve the initial value problem

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

if

$$\Phi = \begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix} e^{2t}$$

is a fundamental matrix.

4. Consider the nonhomogeneous equation

$$\mathbf{x}' = \begin{pmatrix} -5 & 3 \\ 2 & -10 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}.$$

Find the fundamental matrix and its inverse. Find a particular solution to the system and the general solution.

5. Consider the nonhomogeneous equation

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ \cos \omega t \end{pmatrix}, \quad \omega \neq \pm 1.$$

Find the fundamental matrix and its inverse. Find a particular solution to the system and the general solution.

6. Consider the system

$$x' = 3x + 2y + 3, \quad y' = 7x + 5y + 2t.$$

(a) Find the fundamental matrix. **(b)** Use undetermined coefficients to find a particular solution.

7. Find the fundamental matrix for the system

$$x' = x - 3y, \quad y' = 3x + 7y.$$

8. In pharmaceutical studies it is important to model and track concentrations of chemicals and drugs in the blood and in the body tissues. Let x and y denote the amounts (in milligrams) of a certain drug in the blood and in the tissues, respectively. Assume that the drug in the blood is taken up by the tissues at rate $r_1 x$ and is returned to the blood from the tissues at rate $r_2 y$. At the same time the drug amount in the blood is continuously degraded by the liver at rate $r_3 x$. **(a)** Show that the model equations governing the drug amounts in the blood and tissues are

$$x' = -r_1 x - r_3 x + r_2 y,$$
$$y' = r_1 x - r_2 y.$$

(b) Find the critical points in $x, y \geq 0$. **(c)** Find and sketch the nullclines. **(d)** By determining the vector field in regions bounded by the nullclines, argue the $(0, 0)$ is an asymptotically stable node. **(e)** Confirm analytically, confirm that $(0, 0)$ is asymptotically stable. **(f)** In the phase plane sketch the orbit with initial values $x(0) = x_0$, $y(0) = 0$, and interpret the result.

9. In the preceding problem assume that the drug is administered intravenously and continuously at a constant rate D. What are the governing equations in this case? What is the amount of the drug in the tissues after a long time?

10. An animal species of population $P = P(t)$ has a per capita mortality rate m. The animals lay eggs at a rate of b eggs per day, per animal. The eggs hatch at a rate proportional to the number of eggs $E = E(t)$; each hatched egg gives rise to a single new animal.

    a) Write down model equations that govern P and E, and carefully describe the dynamics of the system in the two cases $b > m$ and $b < m$.

    b) Modify the model equations if, at the same time, an egg–eating predator consumes the eggs at a constant rate of r eggs per day.

    c) Solve the model equations in part (b) when $b > m$, and discuss the dynamics.

11. Derive the variation of parameters formula for the second-order equation $x'' + px' + qx = f(t)$ (p, q constant) by transforming the equation to a system and using formula (4.45).

5
Nonlinear Systems

The first three chapters were almost exclusively about linear differential equations, both homogeneous and nonhomogeneous. Such equations have a particularly nice algebraic structure to their solutions, which makes them solvable. Specifically, for both single equations and systems:

- A homogeneous equation has the property that a constant multiple of a solution is a solution and the sum of two solutions is a solution.
- The general solution to a nonhomogeneous equation is the sum of general solution to the homogeneous and a particular solution to the nonhomogeneous equation.
- The general solution to a homogeneous equation with constant coefficients can be found exactly, with a formula.

For nonlinear systems, *none* of these properties is true. Nonlinear equations can rarely be solved by an analytic method. This is unfortunate because Nature often acts in a nonlinear way, and most of the models we face in applications are nonlinear.

In this chapter we take up a study of nonlinear phenomena. In spite of the lack of analytic solutions, we can develop tools that help analyze such problems. The methods are often geometric in nature and they take advantage of the *local* behavior of the equations, which in many cases is *almost* linear. Nonlinear analysis applied to differential equations is one of the most challenging and interesting endeavors in pure and applied science and engineering. The applications in this chapter include nonlinear mechanics, population ecology, epidemiology, chemical reactions, and nonlinear circuits.

5.1 Linearization

For motivation we revisit (see Chapter 1) the ideas of the phase line, equilibria, and stability for the single, nonlinear autonomous equation

$$x' = f(x).$$

An equilibrium $x(t) = x_e$ is a constant solution and therefore satisfies the algebraic equation $f(x_e) = 0$. We also refer to x_e as a critical point. To study the behavior near the equilibrium we impose a small perturbation $u = u(t)$ on the equilibrium state; that is, we move away from the equilibrium a small amount u and then ask if the system returns to equilibrium as time passes, or is equilibrium lost. This is the question of stability of the critical point, or equilibrium state. Quantitatively, we let

$$x(t) = x_e + u(t),$$

or $u(t) = x(t) - x_e$. Because the perturbation u is small, we we can use Taylor's theorem to study how $x(t)$ changes near x_e. In particular, from the differential equation we have $(x_e + u)' = f(x_e + u)$, or

$$u' = f(x_e) + f'(x_e)u + \frac{1}{2}f''(x_e)u^2 + \cdots ,$$

where the dots denote higher-order terms. Neglecting those terms (e.g., if u is small, say 0.1, then u^2 is much smaller, 0.01) we can approximate the last equation by

$$u' = f'(x_e)u,$$

which is a linear differential equation describing how the perturbation $u(t)$ evolves near the equilibrium. The *constant value* $f'(x_e)$, the slope of the tangent line to the graph of $f(x)$ at $x = x_e$, is the *stability indicator*. Clearly the solution to the perturbation equation is

$$u(t) = u(0)e^{f'(x_e)t},$$

so we get the result from Chapter 1:

- If $f'(x_e) < 0$, then small perturbations decay and x_e is a stable equilibrium.
- If $f'(x_e) > 0$, then small perturbations grow and x_e is an unstable equilibrium.

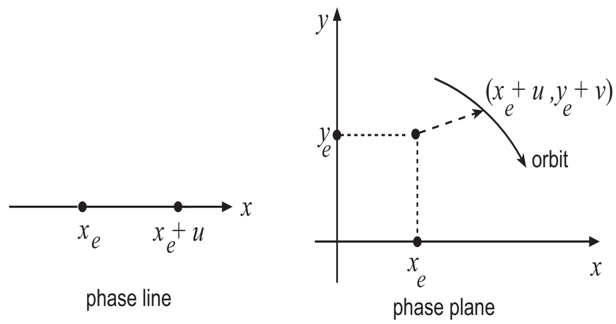

Figure 5.1 (Left) Phase line showing an equilibrium and a small perturbation u from equilibrium. (Right) The phase plane showing a critical point and a small perturbation (u, v). The linearized system dictates how u and v vary near $(0, 0)$, or how x and y vary near (x_e, y_e).

If $f'(x_e) = 0$ then we obtain no information about stability, which forces us to examine higher order terms in the Taylor series. In summary, the local behavior near an equilibrium gives us complete information about the behavior of solutions near that point. In this chapter we take exactly the same approach for nonlinear systems of equations.

We consider the nonlinear system

$$x' = f(x, y), \tag{5.1}$$

$$y' = g(x, y), \tag{5.2}$$

where f and g are continuously differentiable functions. Let $\mathbf{x}^* = (x_e, y_e)$ be an isolated equilibrium, or critical point, which means

$$f(x_e, y_e) = 0, \quad g(x_e, y_e) = 0.$$

Thus the critical points represent constant solutions of the system. Now let $u = u(t)$ and $v = v(t)$ denote small deviations, or **perturbations**, from equilibrium. That is, take

$$x(t) = x_e + u(t), \quad y(t) = y_e + v(t).$$

As in the single equation case, to determine if the perturbations grow or decay we derive differential equations for the perturbations; how those small changes evolve determines the behavior of the orbits near the critical point. Substituting into (5.1)–(5.2) we get the dynamics of the system in terms of $u(t)$ and $v(t)$, or

$$u' = f(x_e + u, y_e + v),$$

$$v' = g(x_e + u, y_e + v).$$

Because the perturbations are small, we can expand the right sides in Taylor series about the critical point (x_e, y_e) to obtain

$$\begin{aligned} u' &= f(x_e, y_e) + f_x(x_e, y_e)u + f_y(x_e, y_e)v + \cdots , \\ v' &= g(x_e, y_e) + g_x(x_e, y_e)u + g_y(x_e, y_e)v + \cdots , \end{aligned}$$

where the dots represent higher-order terms. The first terms on the right sides are zero because (x_e, y_e) is an equilibrium, and the higher-order terms are small in comparison to the linear terms. Therefore the perturbation equations can be approximated locally, or near the critical point, by the *linear* system

$$\begin{aligned} u' &= f_x(x_e, y_e)u + f_y(x_e, y_e)v, \\ v' &= g_x(x_e, y_e)u + g_y(x_e, y_e)v, \end{aligned}$$

near $(u, v) = (0, 0)$. This system for the small perturbations, or changes, is called the linearized perturbation equations, or simply the **linearization** of (5.1)–(5.2) at the critical point (x_e, y_e). The linearization has a critical point at the origin $(0, 0)$ corresponding to the critical point (x_e, y_e) for the nonlinear system. In matrix form we can write the linearization as

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} f_x(x_e, y_e) & f_y(x_e, y_e) \\ g_x(x_e, y_e) & g_y(x_e, y_e) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{5.3}$$

or, in concise notation,

$$\mathbf{z}' = J\mathbf{z},$$

where $\mathbf{z}(t) = (u(t), v(t))^{\mathrm{T}}$. The matrix $J = J(x_e, y_e)$ of first partial derivatives of f and g defined by

$$J(x_e, y_e) = \begin{pmatrix} f_x(x_e, y_e) & f_y(x_e, y_e) \\ g_x(x_e, y_e) & g_y(x_e, y_e) \end{pmatrix} \tag{5.4}$$

is called the **Jacobian matrix** at the equilibrium (x_e, y_e). Note that the entries of this matrix are *constants* because the partial derivatives are evaluated at the equilibrium. We assume that the matrix J does not have a zero eigenvalue, i.e., $\det J \neq 0$. If $\det J = 0$, we must analyze the higher-order terms in the Taylor expansions of the right sides of the equations.

We already know that the nature of the critical point of the linearization (5.3) is determined entirely by the eigenvalues of the Jacobian matrix J. Does this imply anything about the nonlinear system? The answer is yes, almost all of the time.

There is an important theorem in dynamics that states that the nonlinear close to the critical point (x_e, y_e) behaves much like the linearized system at the origin $(0, 0)$. The only restriction is that the *linearized system cannot have eigenvalues that are zero or have zero real parts.* In one of these exceptional cases the linearized system can have a center, but the nonlinear system has a spiral.

Pictorially, in the non-exceptional cases, the nonlinear system has a slightly distorted phase diagram from that of its linearization; the distortion is caused by the fact that small terms were deleted to obtain the linearization. Rather than formally state a technical theorem, we summarize the key results as follows.

Behavior near a critical point:

- If $(0, 0)$ is asymptotically stable for the linearization (5.3), then the perturbations decay and (x_e, y_e) is asymptotically stable for the nonlinear system (5.1)–(5.2). This occurs when J has negative eigenvalues, or complex eigenvalues with negative real part. The conditions for **asymptotic stability** are

$$\operatorname{tr} J(x_e, y_e) < 0 \quad \text{and} \quad \det J(x_e, y_e) > 0. \tag{5.5}$$

  We use this result often in analyzing nonlinear systems.

- If $(0, 0)$ is unstable for the linearization (5.3), then some or all of the perturbations grow and (x_e, y_e) is unstable for the nonlinear system (5.1)–(5.2). This occurs when J has a positive eigenvalue or complex eigenvalues with positive real part.

- The exceptional case for stability is that of a center. If $(0, 0)$ is a center for the linearization (5.3), then (x_e, y_e) may be a center or a spiral (stable or unstable) for the nonlinear system (5.1)–(5.2). This case occurs when J has purely imaginary eigenvalues.

- In the borderline case of equal eigenvalues, the nonlinear system maintains its stability or instability, but the local behavior of orbits may change because of inclusion of the nonlinear terms. For example, a node for the linearized system may change into spiral for the nonlinear system, but stability will not be affected.

Most of the time we are only interested in whether an equilibrium is asymptotically stable or unstable. This can be determined by examining the trace of J and the determinant of J at the equilibrium, as stated in condition (5.5).

Nullclines. To sketch the phase diagram, it is important to plot the set of points where the vector field is *vertical*; this is the set of points (x, y) where

$$x' = f(x, y) = 0 \qquad (x \text{ nullclines}).$$

The curves where this occurs are called x **nullclines**. The y **nullclines** are the the points where the vector field is *horizontal*, or

$$y' = g(x, y) = 0 \qquad (y \text{ nullclines}).$$

Observe that x and y nullclines intersect at a critical point, where the vector field vanishes.

One can use computer algebra systems or calculators to draw phase plane diagrams. With computer algebra systems there are two options. You can write a program, or code, to numerically solve and plot the solutions, or you can use their built-in programs that plot solutions automatically. See Chapter 6.

Sketching the phase diagram

In summary, we have developed a set of tools to analyze nonlinear systems. We can systematically follow the steps below to obtain a complete phase diagram.

1. Find the critical points and check their type and stability by examining the eigenvalues of the Jacobian matrix J of the linearized system.
2. Draw the x and y nullclines.
3. Using the differential equations, find the direction of the vector field in the regions bounded by the nullclines. The directions can be described by actual geographical directions: NE, SE, SW, NW.
4. Sketch in representative orbits from the information above.
5. To get more detail, find directions of the separatrices (if any) at equilibria; these directions are indicated by the direction of the eigenvectors of the linearization J. Both saddle points and nodes have separatrices.
6. In some simple cases, divide the equations and integrate to find the orbits in terms of x and y (this may be impossible in most cases).
7. Use a software package or graphing calculator to obtain an accurate phase diagram with actual orbits.

Example 5.1

Consider the system

$$x' = f(x, y) = -x + xy \tag{5.6}$$

$$y' = g(x, y) = -4y + 8xy. \tag{5.7}$$

To find the critical points we solve the two equations

$$-x + xy = 0, \quad -4y + 8xy = 0,$$

simultaneously. Both factor and we obtain

$$x(-1 + y) = 0, \quad 4y(-1 + 2x) = 0.$$

The first equation gives $x = 0$ or $y = 1$. This give two cases. Substituting $x = 0$ into the second equation we get $y = 0$. Therefore $x = 0,\ y = 0$ is a critical point. Next, if $y = 1$, then the second equation gives $x = \frac{1}{2}$, leading to the critical point $x = \frac{1}{2}$, $y = 1$. There are two critical points,

$$\text{critical points: } (0,0), \quad \left(\frac{1}{2}, 1\right).$$

Now we calculate the Jacobian matrix at each point. In general,

$$J(x,y) = \begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix} = \begin{pmatrix} -1+y & x \\ 8y & -4+x \end{pmatrix}.$$

Evaluating at the critical point $(0,0)$,

$$J(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}.$$

This is a diagonal matrix so its eigenvalues are on the diagonal: $\lambda = -1,\ -4$. [Or, we could solve the characteristic equation, $\lambda^2 + 5\lambda + 4 = 0$.] Therefore the critical point $(0,0)$ is an *asymptotically stable node*. Evaluating J at the critical point $(\frac{1}{2}, 1)$, we get

$$J\left(\frac{1}{2}, 1\right) = \begin{pmatrix} 0 & \frac{1}{2} \\ 8 & 0 \end{pmatrix}.$$

The characteristic equation is $\lambda^2 - 4 = 0$, giving eigenvalues $\lambda = \pm 2$. These are real with opposite signs, so the critical point $(\frac{1}{2}, 1)$ is an *unstable saddle point*.

There is often a lot of flexibility in sketching the phase diagram. We could find the eigenvector directions for each eigenvalue; these indicate the directions of the separatrices. Or, we can proceed graphically by sketching the nullclines and direction field.

Let us do the latter. Refer to Figure 5.2. The x-nullclines are where $f(x,y) = x(-1+y) = 0$, or along the two lines $x = 0$ and $y = 1$. Along those lines the vector field points up or down. The y-nullclines are where $g(x,y) = 4y(-1+2x) = 0$, or along the lines $y = 0$ and $x = \frac{1}{2}$. Along these lines the vector field is left or right. Observe, the intersections of the vertical and horizontal nullclines are critical points.

Next let's put directions on the vector field along the nullclines. On the x nullcline (y axis) we have $x' = 0$, $y' = -4y$. So $y' < 0$ when $y > 0$, and $y' > 0$ when $y < 0$. So, on the positive y axis the arrows are down, and on the negative y axis the arrows are up. This means that both the positive and negative y axes are orbits approaching the origin. A similar argument can be made on the x axis, which is a y nullcline; it shows that the negative x axis is an orbit that enters the origin from the left and the positive x axis is an orbit

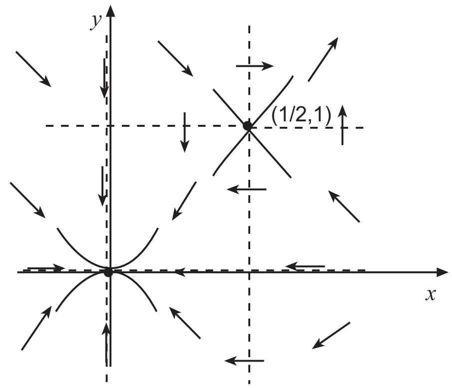

Figure 5.2 First steps in constructing a phase diagram. Locate the critical points, which are at intersections of nullclines; use the Jacobian matrix to determine the type and stability of the critical points; put directions on the nullclines and in the regions bounded by the nullclines.

entering the origin from the right. The directions along the other nullclines are easily calculated and are shown it Figure 5.2.

The next task it to compute the direction field in the regions bounded by all the nullclines. For example, at $(1,-1)$, a point in the 4th quadrant, the vector field evaluates to $x' = -2$, $y' = -4$, which is SW, as indicated. We can continue this process with the other regions, but because we know $(0,0)$ is a stable node and $(\frac{1}{2},1)$ is a saddle, we can easily guess those directions to get a consistent diagram, noting that orbits cannot crash into each other, and they cannot change direction unless they cross a nullcline. Moreover, we know that a saddle point has separatrices in the direction of the eigenvectors of J at that point, and we can approximate where they fit. Recall that for linear systems separatrices are linear orbits; for nonlinear systems they are distorted. We leave it to the reader to sketch in some key orbits. □

Example 5.2

Consider the decoupled nonlinear system

$$x' = x - x^3, \quad y' = 2y.$$

The equilibria are $(0,0)$ and $(\pm 1, 0)$. The Jacobian matrix at an arbitrary (x,y) is

$$J(x,y) = \begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix} = \begin{pmatrix} 1-3x^2 & 0 \\ 0 & 2 \end{pmatrix}.$$

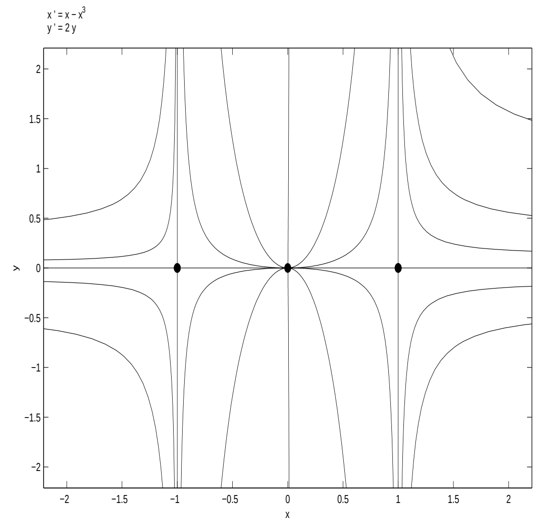

Figure 5.3 Phase diagram for the system $x' = x - x^3,\ y' = 2y$ generated by software. In the upper half-plane the orbits are moving upward, and in the lower half-plane they are moving downward.

Therefore

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

which has eigenvalues 1 and 2. Thus $(0,0)$ is an unstable node. Next

$$J(1,0) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}, \quad J(-1,0) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix},$$

and both have eigenvalues -2 and 2. Therefore $(1,0)$ and $(-1,0)$ are saddle points. The phase diagram is easy to draw. The vertical (where $x' = 0$) nullclines are $x = 0$, $x = 1$, and $x = -1$, and the horizontal (where $y' = 0$) nullcline is $y = 0$, or the x-axis. Along the x axis we have $x' > 0$ if $-1 < x < 1$, and $x' < 0$ if $|x| > 1$. The phase portrait is shown in Figure 5.3. □

Example 5.3

Consider the simple nonlinear system

$$x' = y^2, \tag{5.8}$$

$$y' = -\frac{2}{3}x. \tag{5.9}$$

Clearly, the origin $x = 0$, $y = 0$, is the only critical point. The Jacobian matrix at the origin is easily

$$J(0,0) = \begin{pmatrix} 0 & 0 \\ \frac{2}{3} & 0 \end{pmatrix}.$$

This has determinant equal to zero, and both eigenvalues are zero. The linearization procedure *does not apply*, so we proceed with a different strategy.

In this case we can divide the two equations and separate variables to get

$$3y^2 dy = -2x\, dx.$$

Integrating gives

$$y^3 = -x^2 + C.$$

Rearranging,

$$y = (C - x^2)^{1/3}.$$

We have obtained the orbits for system (5.8)–(5.9) in terms of x and y, losing the dependence on time. These are easily plotted, e.g., on a calculator, for different values of C, and several are shown in Figure 5.4. Note that the x axis ($y = 0$) is a x-nullcline (vertical vector field) and the y axis ($x = 0$) is a y-nullcline (horizontal vector field). It is important to notice that the critical point is *not* of the usual type (node, saddle, etc.). Unusual orbital behavior is common for nonlinear systems.

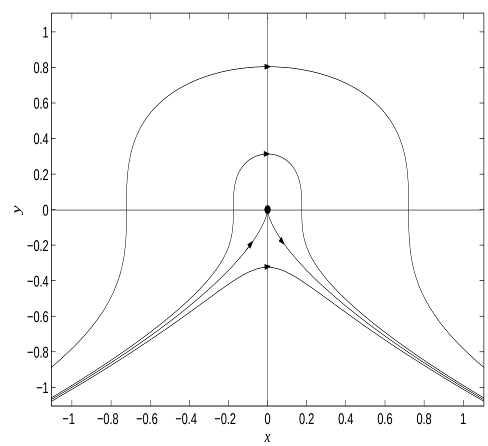

Figure 5.4 Phase diagram for $x' = y^2$, $y' = -\frac{2}{3}x$. Because $x' > 0$, all the orbits are moving to the right as time increases. The critical point is not of standard type.

If we want orbits in terms of time t, we can write (5.8) as

$$x' = y^2 = (C - x^2)^{2/3},$$

which is a single differential equation for $x = x(t)$. We can separate variables, but the result is not very satisfying because we get a complicated integral. Components of solutions are not easily obtained for nonlinear problems. However, the qualitative behavior shown in the phase diagram is often all we want. If we need time series plots, we can obtain them using a numerical method. □

EXERCISES

1. For each of the following nonlinear systems: **(a)** Find all the critical points. **(b)** Use the Jacobian matrix to classify the critical points (saddle, node, etc., stable, asymptotically stable, etc.) **(c)** In the phase plane sketch the nullclines and indicate the direction field in regions bounded by the nullclines. **(d)** Insofar as possible, sketch a rough phase diagram indicating several key orbits.

   a) $x' = x - y,\ y' = 2x(-1 + y)$.
   b) $x' = x - xy^2,\ y' = x + y$.
   c) $x' = 1 - x^2,\ y' = y + 2$.
   d) $x' = -x + xy,\ y' = -2y(1 - 4x)$.
   e) $x' = x - y + x^2,\ y' = x + y$.
   f) $-x - 2y,\ y' = 2x - y + xy^2$.

2. Consider the system

$$x' = x - xy + hx^2,$$
$$y' = -y(1 - x),$$

   where h is a given parameter. For the following 3 values of h find the critical points and classify them as to type and stability: $h = -1,\ 1,\ 8$.

3. Consider the system

$$x' = -1/y, \quad y' = 2x.$$

   a) Are there any equilibrium solutions?
   b) Sketch the nullclines and vector field.
   c) Find a relationship between x and y that holds on orbits, and plot several orbits in the phase plane.

4. Consider the nonlinear system $x' = x^2 + y^2 - 4,\ y' = y - 2x$.

   a) Find the two equilibria and plot them in the phase plane.

b) On the plot in part (a), sketch the nullclines.

c) Indicate the direction of the vector field in the regions separated by nullclines. Can you determine the nature (node, center, etc.) and stability of the equilibria only from this information?

5. Repeat parts (a), (b), and (c) of the previous problems for the nonlinear system $x' = y + 1$, $y' = y + x^2$.

6. Find all equilibria for the system $x' = \sin y$, $y' = 2x$.

7. Consider the nonlinear system $x' = x^2 - y^2$, $y' = x - y$.

a) Find and plot the equilibria in the phase plane. Are they isolated?

b) Show that orbits are given by $x + y + 1 = Ce^y$, where C is a constant. Plot a few of these orbits. Hint: Determine dy/dx.

c) Sketch nullclines and the vector field in regions bounded by the nullclines.

d) Describe the fate of the orbit that begins at $(\frac{1}{4}, 0)$ at $t = 0$ as $t \to +\infty$ and as $t \to -\infty$.

e) Draw a phase plane diagram, being sure to indicate the directions of the orbits.

8. Find the equation of the orbits of the system $x' = e^x - 1, \;\; y' = ye^x$ and plot the the orbits in phase plane.

9. Write down an equation for the orbits of the system $x' = y, \;\; y' = 2y + xy$. Sketch the phase diagram.

10. For the following system find the equilibria, sketch the nullclines and the direction of the flow along the nullclines, and sketch the phase diagram:

$$x' = y - x^2, \;\; y' = 2x - y.$$

What happens to the orbit beginning at $(1, 3/2)$ as $t \to +\infty$?

11. Consider the system

$$x' = 2x(1 - \frac{x}{2}) - xy,$$
$$y' = y\left(\frac{9}{4} - y^2\right) - x^2 y.$$

Find the equilibria and sketch the nullclines. Use the Jacobian matrix to determine the type and stability of each equilibrium point and sketch the phase diagram.

12. Completely analyze the nonlinear system

$$x' = y, \quad y' = x^2 - 1 - y.$$

13. Consider the system

$$x' = xy - 2x^2$$
$$y' = x^2 - y.$$

Find the equilibria and use the Jacobian matrix to determine their types and stability. Draw the nullclines and indicate on those lines the direction of the vector field. Draw a phase diagram.

14. Show that the origin is asymptotically stable for the system

$$x' = y,$$
$$y' = 2y(x^2 - 1) - x.$$

15. Consider the system

$$x' = y,$$
$$y' = -x - y^3.$$

Show that the origin for the linearized system is a center, while for the nonlinear system the origin is asymptotically stable. Hint: Show that $(d/dt)(x^2 + y^2) < 0$.

5.2 Nonlinear Mechanics

In this section we take up one of the most important and familiar applications of nonlinear systems theory, namely nonlinear mechanics. The techniques in the previous section apply, and we introduce new insights that yield the phase diagram with little effort, especially for conservative systems.

The general form of Newton's law for a particle of mass m moving in on a straight line, the x axis, under the influence of an external force F is

$$mx'' = F(x, x'), \tag{5.10}$$

where $x = x(t)$ is the position of the particle. We assume the force does not depend explicitly on time, which makes the problem autonomous. If we let $y = x'$ denote the velocity of the particle, then we immediately get the system

$$x' = y, \tag{5.11}$$

$$y' = \frac{1}{m}F(x, y). \tag{5.12}$$

We can make some instant conclusions regarding equilibria. For the system (5.11–5.12) the critical points are determined by solving the simultaneous equations

$$y = 0, \quad F(x, 0) = 0.$$

Hence, they are points $(x_e, 0)$ on the x axis in the phase plane where $F(x_e, 0) = 0$. Clearly, an equilibrium occurs if the force on the particle is zero, with zero velocity. In the phase plane the x component of the vector field satisfies $x' = y$, so the particle moves to the right in the upper half plane, and to the left in the lower half-plane. The x axis is an x-nullcline with direction up or down. All vertical lines $x = x_e$ are y nullclines with direction to the right in the upper half plane, and to the left in the lower half plane. Therefore, much of the phase diagram is known, and all that remains is to find the nature and stability of the equilibria, and the linearization procedure can often resolve that.

Conservative forces, where F depends only on x, are the simplest and there is a particularly easy way to plot the phase plane orbits from the conservation of energy equation.

Example 5.4

(**Conservative systems**) We recall from calculus that $F = F(x)$ is a **conservative force** if there exists a **potential function** $V = V(x)$ for which[1]

$$F(x) = -\frac{dV}{dx}.$$

In words, *the force is the negative gradient of the potential*. Therefore the potential is given by

$$V(x) = -\int F(x)dx + C.$$

The arbitrary constant C is often fixed so that the potential at $x = 0$ is zero, or $V(0) = 0$, or sometimes $V(+\infty) = 0$.

If we divide the governing equations, (5.11–5.12) and separate variables, we obtain

$$my\,dy = F(x)dx.$$

Integrating both sides leads to

$$\frac{1}{2}my^2 = \int F(x)dx + E = -V(x) + E,$$

where E is a constant of integration, denoting total energy. Therefore

$$\frac{1}{2}my^2 + V(x) = E, \tag{5.13}$$

[1] Occasionally we write dV/dx as $V'(x)$. The "prime" is understood as an x derivative because that is the independent variable; note that x' means a time derivative.

which is the **energy conservation law**: *the kinetic plus potential energy for a conservative system is constant*. The constant E represents the total energy in the system, and it can be computed from knowledge of the initial position $x(0) = x_0$ and initial velocity $y(0) = y_0$; that is,

$$E = \frac{1}{2}y_0^2 + V(x_0). \quad \square$$

The conservation of energy law is a reduction of Newton's second law; the latter is a second-order equation, whereas (5.13) is a first-order equation. If we replace the position y by dx/dt it may be recast into

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - V(x)}. \tag{5.14}$$

This equation is separable, and its solution defines the displacement $x = x(t)$ implicitly. The appropriate sign ($\pm$) is taken depending on whether the velocity is positive or negative during a certain phase of the motion. Integrating (5.14) gives the implicit solution

$$t = \pm\sqrt{\frac{m}{2}}\int \frac{dx}{\sqrt{E - V(x)}} + C. \tag{5.15}$$

For oscillatory motion, this formula is useful in obtaining an expression for the period of the motion by integrating over some portion of a period, depending on the symmetry of the orbit.

We can analyze conservative systems qualitatively and simply in the xy phase plane by plotting y versus x from equation (5.13) for different values of the parameter E. This one-parameter family of curves are the orbits of the system, each one representing a constant energy curve. Specifying the initial condition determines the E value and picks out the curve with that total energy.

Example 5.5

(**Oscillator**) Consider a spring–mass system without damping. The governing equation is

$$mx'' = -kx,$$

where k is the spring constant. The force is $-kx$ and the potential energy $V(x)$ is given by

$$V(x) = -\int -kx\,dx = \frac{k}{2}x^2.$$

We have picked the constant of integration to be zero, which automatically sets the zero level of potential energy at $x = 0$ (i.e., $V(0) = 0$). Conservation of energy is expressed by (5.13), or

$$\frac{1}{2}my^2 + \frac{k}{2}x^2 = E,$$

which plots as a family of concentric ellipses in the xy phase plane, one ellipse for each value of E. See Figure 5.5. Each curve, along which energy is conserved,

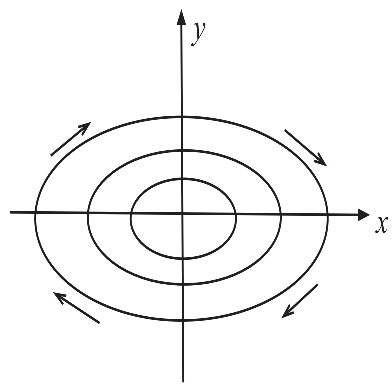

Figure 5.5 Elliptical orbits $\frac{1}{2}my^2+\frac{k}{2}x^2 = E$ for different values of the energy E. The direction of the orbits in each quadrant is shown; they are traced out clockwise because $x' = y > 0$, or x increases in the upper half plane. Similarly, x decreases along the orbit in the lower half-plane.

represents an oscillation, and the mass tracks on one of these orbits in the phase plane, continually cycling as time passes, in the clockwise direction. This is because $x' = y$, so x increases when y is positive (in the upper half-plane) and decreases when y is negative (in the lower half-plane). Clearly, the position and velocity cycle back and forth.

Generally, the conservation of energy curves (orbits) can be obtained graphically in a simple way. Solving the conservation equation

$$\frac{1}{2}my^2 + \frac{k}{2}x^2 = E$$

for y gives

$$y = \pm\sqrt{\frac{2}{m}}\sqrt{E - \frac{1}{2}kx^2}.$$

Then, a calculator can be used for the plot. Or, it can be found graphically as follows. Simply plot the potential energy function $V(x) = \frac{1}{2}kx^2$ and line $y = E$ of constant energy on the same axes (energy vs. x); then subtract graphically the square root of the difference $E - V(x)$ of the two to obtain the shape of the upper branch of the elliptical orbit, which is plotted in xy phase space. Reflect that half orbit through the x axis to get the lower branch of the ellipse. Figure 5.6 illustrates this procedure. To obtain additional orbits, take different constant energy levels E.

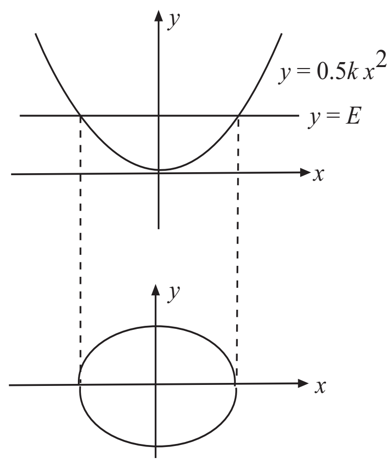

Figure 5.6 Graphical method to plot $y = \sqrt{2/m}\sqrt{E - kx^2/2}$. Plot $y = E$ and $y = \frac{1}{2}kx^2$ (top plot), and approximately subtract the square root of the difference, multiplied by the factor $\sqrt{2/m}$, to get the upper branch of the ellipse (lower plot). Reflect this curve through the x axis to get the lower portion of the ellipse. The orbit is counterclockwise since $x' = y > 0$ in the upper half plane.

To compute the period T we use (5.15) and integrate over one-fourth of a period along the orbit from a point $(0, y_M)$ to the point $(x_M, 0)$ through the first quadrant to get

$$\frac{T}{4} = \sqrt{\frac{m}{2}} \int_0^{x_M} \frac{dx}{\sqrt{E - \frac{1}{2}kx^2}}.$$

Notice that if y_M and E are given then x_M is determined by the conservation law. Then this integral can be computed exactly using elementary calculus. We leave this calculation to the reader. □

Example 5.6

If a particle of mass $m = 1$ moves on an x-axis under the influence of a nonlinear force $F(x) = 3x^2 - 1$, then the equations of motion are

$$x' = y,$$
$$y' = 3x^2 - 1.$$

The force depends only on x and thus there is a potential energy function given by

$$V(x) = -\int (3x^2 - 1)dx = -x^3 + x = x(1-x)(1+x).$$

We have taken $V(0) = 0$. Therefore, conservation of energy is

$$\frac{1}{2}y^2 + x(1-x)(1+x) = E.$$

Solving for the velocity y gives

$$y = \pm\sqrt{2}\sqrt{E - x(1-x)(1+x)}. \tag{5.16}$$

The total energy E can be written in terms of the initial position and velocity as $E = \frac{1}{2}y^2(0) + (-x(0)^3 + x(0))$. We can use (5.16) to plot orbits using a calculator or computer algebra system.

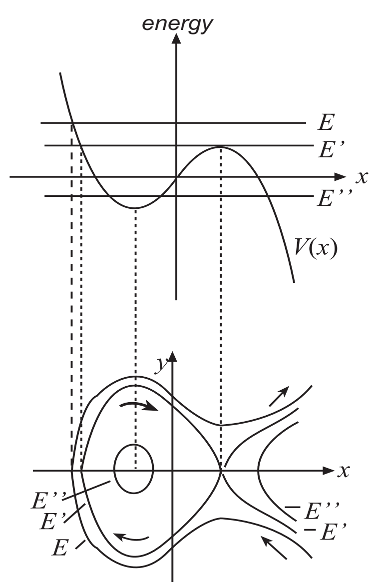

Figure 5.7 A rough geometrical construction of orbits using conservation of energy, given the three total energies E, E', and E'' of the system. Compare to the exact, computed orbits in Figure 5.8.

However, we again illustrate the geometric technique using (5.16) to sketch approximate orbits. Refer to Figure 5.7. The potential energy $V(x)$ is plotted as shown and we choose an energy level E. We subtract $E - V(x)$ graphically, and then approximately plot $y = \sqrt{2}\sqrt{E - V(x)}$. This is labeled on the lower panel of the plot; we reflected it through the x axis because of the $\pm$ sign. Thus we obtain a single non-periodic orbit as shown, and it is directed to the right for $y > 0$ and to the left for $y < 0$. Next, we make a similar plot for another special value of energy E' where the maximum potential energy occurs. The resulting orbit is in two parts—the closed orbit and the two separatrices, as shown. For a still smaller value of energy E'' we obtain a periodic orbit indicating a *potential well* at the minimum of $V(x)$; to the right is a hyperbolic type orbit extending from infinity in the 4th quadrant to infinity in the 1st quadrant, both approaching the separatrices. So this system can exist in one of two different configurations with the same energy, depending on the initial condition. We can begin to see that stable center is positioned at the minimum potential energy, and a saddle is appearing at the maximum potential energy. This geometrical method is sometimes crude, but it can give us a good idea of the dynamics.

Now let us test the mathematics to verify our intuition. The critical points are easily

$$\left(\sqrt{\frac{1}{3}}, 0\right), \quad \left(-\sqrt{\frac{1}{3}}, 0\right).$$

The Jacobian matrix is

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ 6x & 0 \end{pmatrix}.$$

If $x = \sqrt{1/3}$ and $y = 0$ then the eigenvalues $\lambda = \pm 6/\sqrt{3}$ (real and opposite sign), which signifies a saddle point. If $x = -\sqrt{1/3}$ and $y = 0$ then the eigenvalues are $\lambda = \pm 6i/\sqrt{3}$ (purely imaginary), which is the exceptional case. However, the geometrical argument given above shows that $(-\sqrt{1/3}, 0)$ is indeed a center.

Figure 5.8 shows several orbits computed by software. Let us discuss their features. There are two points, $x = \sqrt{1/3}$, $y = 0$ and $x = -\sqrt{1/3}$, $y = 0$, where $x' = y' = 0$. These are two equilibrium solutions where the velocity is zero and the force is zero (so the particle cannot be in motion). These are the points where the nullclines cross. The x nullcline is $x' = y = 0$, or the y axis; there the orbits are vertical. The y nullclines are $y' = 3x^2 - 1 = 0$, or the lines $x = \pm\sqrt{1/3}$. There, the orbits cross horizontally. The equilibrium solution $x = -\sqrt{1/3}$, $y = 0$ has the structure of a center, and for initial values close to this critical the system will oscillate. The other equilibrium $x = \sqrt{1/3}$, $y = 0$ has the structure of a saddle point. Because $x' = y$, for $y > 0$ we have $x' > 0$, and the orbits are directed to the right in the upper half-plane. For $y < 0$ we have $x' < 0$, and the orbits are directed to the left in the lower half-plane. For large initial energies the system does not oscillate but rather goes to $x = +\infty$, $y = +\infty$; that is, the mass moves farther and farther to the right with faster speed. ☐

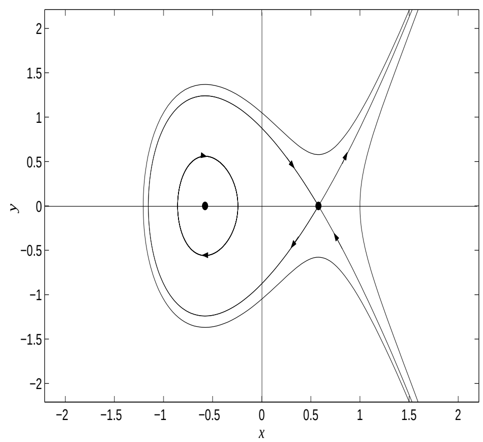

Figure 5.8 Plots of the constant energy curves $\frac{1}{2}y^2+x(1-x)(1+x) = E$ in the xy-phase plane. These curves are the orbits and show how position and velocity relate. Because $x' = y$, the orbits are moving to the right (x is increasing) in the upper half-plane $y > 0$, and to the left (x is decreasing) in the lower half-plane $y < 0$. This plot was produced by MATLAB®.

Example 5.7

(**Pendulum**) Determining the motion of a pendulum is a classical problem in mechanics. This example shows different ways to proceed. Consider a frictionless pendulum of length l whose bob has mass m (Figure 5.9). As a state variable we choose the angle $\theta = \theta(t)$ that the pendulum makes with the vertical. As time passes, the bob goes back and forth and traces out an arc on a circle of radius l. Let s denote the arclength measured from rest ($\theta = 0$) along the arc. By simple geometry $s = l\theta$. As the bob moves, its kinetic energy is one-half its mass times the velocity squared, or

$$\text{KE } = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = \frac{1}{2}ml^2(\theta')^2.$$

Its potential energy is mgh, where h is the height above the zero-potential energy level, taken where the pendulum is at rest. Therefore the potential energy is

$$\text{PE} = mgl(1 - \cos\theta).$$

Energy is conserved, so

$$\frac{1}{2}ml^2(\theta')^2 + mgl(1 - \cos\theta) = E, \tag{5.17}$$

where E is the constant energy. If we displace the bob and release it, the initial conditions are $\theta(0) = \theta_0$ and $\theta'(0) = 0$. Using (5.17) we can calculate the total energy in this configuration. Using the variable $\omega(t)$ to denote the *angular*

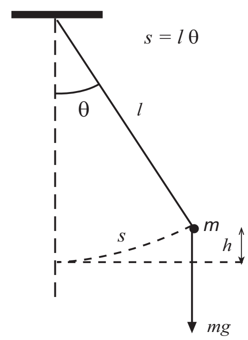

Figure 5.9 A pendulum consisting of a mass m attached to a rigid weightless rod of length l. The force of gravity is mg, directed downward. The potential energy is mgh where h is the height of the mass above the equilibrium position. The kinetic energy is taken along the path of motion, the arc. The arc length is $s = l\theta$ and the velocity is $s' = l\theta'$. So the kinetic energy is $(m/2)l^2(\theta')^2$.

velocity,

$$\omega(t) = \theta'(t),$$

we can write the conservation law as

$$\omega = \pm\sqrt{\frac{2}{ml^2}}\sqrt{E - mgl(1 - \cos\theta)}. \tag{5.18}$$

We can plot orbits in the $\theta\omega$ phase plane by the graphical method illustrated in the last example. We plot energy E and potential energy on the same coordinate axes, energy vs. θ, and graphically subtract to get the orbits in the $\theta\omega$ phase plane.

We can use (5.18) to find $\theta(t)$ implicitly by separating variables and integrating; we obtain

$$t = \pm\sqrt{\frac{ml^2}{2}}\int \frac{d\theta}{\sqrt{E - mgl(1 - \cos\theta)}} + C.$$

The integral on the right cannot be found in terms of standard functions, but rather in terms of special functions called *elliptic functions*. These are cataloged in tables and software. We leave it to the reader to explore this topic. If we

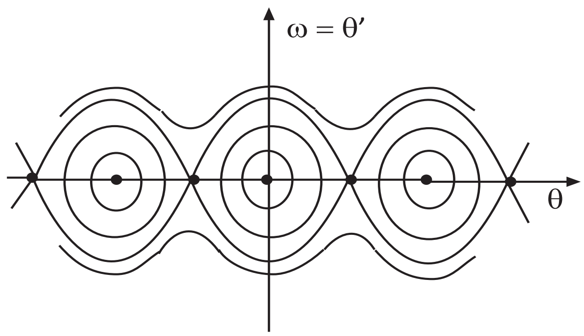

Figure 5.10 Phase-plane diagram for the pendulum. The critical points are $(n\pi, 0)$, which are saddles for n odd, and centers for n even. The oscillations are clockwise because $\theta' = \omega > 0$ for $\omega > 0$.

differentiate the conservation law (5.17) with respect to t and use the chain rule, we get

$$\theta'' + \frac{g}{l}\sin\theta = 0, \tag{5.19}$$

which is a second-order nonlinear DE in $\theta(t)$ called the **pendulum equation**. It is Newton's second law of motion. It can also be derived *directly* by determining the forces on the bob, which we leave as an exercise. We can clearly write (5.19) as a system

$$\begin{aligned}\theta' &= \omega,\\ \omega' &= -\frac{g}{l}\sin\theta.\end{aligned}$$

The critical points are $(\theta_e, \omega_e) = (n\pi, 0)$, $n = 0, \pm 1, \pm 2, \ldots$. The Jacobian matrix at these points is therefore

$$J(n\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos n\pi & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{g}{l}(-1)^{n+1} & 0 \end{pmatrix}.$$

This implies that the eigenvalues of J are $\lambda = \pm\sqrt{g/l}$ for n odd, and therefore the critical points $(n\pi, 0)$ are saddle points; if n is even, then the eigenvalues are purely imaginary, $\lambda = \pm i\sqrt{g/l}$, and the linearized system has a center at $(n\pi, 0)$. This latter case is the exceptional case and we cannot conclude automatically that the even critical points are centers for the nonlinear system.

As it turns out, they are in fact centers, as the geometrical argument above shows. The phase plane diagram (Figure 5.10) shows the critical points for $n = -2, -1, 0, 1, 2$ with the saddle and center structures. For states θ between $-\pi$ and π, only the middle portion of the phase plot with a center is applicable; the states $\theta = \pi$ or $\theta = -\pi$ represent cases where the pendulum is inverted, which are clearly unstable states (the saddles). The curves connecting the saddle points are separatrices. For large energies, the pendulum will continually spin around, represented by the top and bottom orbits in the diagram. □

EXERCISES

1. Formulate the second-order nonlinear equation

$$x'' + x^2x' - x = 0$$

as a system of two first-order equations. Find the critical points and classify them as to type and stability.

2. Consider a dynamical system governed by the equation $x'' = -x + x^3$, with mass $m = 1$.

   a) Find the potential energy $V(x)$ with $V(0) = 0$.

   b) What is the total energy E is in the system if $x(0) = 2$ and $x'(0) = 1$?

   c) Using the conservation law, plot the orbit in the xy phase plane of a particle having this amount of total energy. Indicate by arrows the direction that this orbit is traced out as time increases.

3. Consider a dynamical system governed by the equation $x'' = -x^2$.

   a) Find the potential energy $V(x)$ with $V(0) = 0$ and write the conservation of energy law.

   b) Determine the total energy E of the system if $x(0) = 1$ and $x'(0) = 0$.

   c) Plot the orbit in the xy phase plane of a particle having this amount of total energy and indicate by arrows the direction that the orbit is traced out as time increases.

   d) Explain in words how the system changes as time increases.

4. In the preceding exercise, find an implicit formula for the solution $x = x(t)$ of the DE and initial conditions.

5. In a nonlinear spring–mass system the equation governing the position is $x'' = -2x^3$.

a) Show that conservation of energy for the system can be expressed as $y^2 = C - x^4$, where C is a constant.

b) Plot this set of orbits in the phase plane for different values of C.

c) If $x(0) = x_0 > 0$ and $x'(0) = 0$, show that the period of the oscillations is

$$T = \frac{4}{x_0} \int_0^1 \frac{dr}{\sqrt{1 - r^4}}.$$

Sketch a graph of the period T versus the initial position x_0.

6. Consider a conservative system with mass $m = 2$ whose potential energy is

$$V(x) = (x+1)^2(x-2)^2.$$

a) What is the force?

b) Using graphical techniques described in this section, draw the curves (orbits) in xy phase space of constant energy for the case $E = 1, 2, 3$.

c) As time increases, indicate the direction of motion on the orbits.

d) If the mass starts at $x = 0$, $y = 3$ at $t = 0$, describe how its position $x = x(t)$ changes over time. Estimate its maximum positive and negative displacement from $x = 0$.

7. Newton's law of gravity states that the equation of motion for a mass m in the gravitational field of the earth of mass M is

$$mx'' = -\frac{GMm}{(x+R)^2},$$

where R is the radius of the earth, x is the height of the object above the surface of the earth, and G is the universal gravitational constant. The force on the right side is the *inverse-square law*. At the surface the force is $-mg$, where g is the gravitational acceleration at sea level. Thus, $GMm/R^2 = mg$ and we can write

$$x'' = -\frac{gR^2}{(x+R)^2}.$$

Initially, assume that $x(0) = 0$ and $x'(0) = y_0$, where y_0 is the initial velocity.

a) Find the potential energy function $V(x)$ assuming $V(0) = 0$, and write down the conservation of energy law.

b) Show that the velocity is given by

$$y = \pm\sqrt{y_0^2 - 2gR\left(1 - \frac{R}{x+R}\right)}.$$

c) Using the graphical techniques described in this section, sketch three orbits in phase space for the cases $y_0 > \sqrt{2gR}$, $y_0 = \sqrt{2gR}$, and $y_0 < \sqrt{2gR}$. Explain these orbits and state why is $\sqrt{2gR}$ called the *escape velocity*?

d) Show that $\sqrt{2gR}$ is approximately 11.1 km/sec. Give the answer in miles per hour.

8. Derive the pendulum equation (5.19) from the conservation of energy law (5.17). Hint: Take the derivative with respect to t, using the chain rule.

9. Derive the pendulum equation (5.19) from Newton's second law of motion by finding the force on the bob in an arbitrary position on its arc of motion. Hint: Use arclength $s = s(t)$ measured from rest ($\theta = 0$) as the position coordinate and observe that the downward force mg can be resolved into a force tangential to the path and a force normal (perpendicular) to the path.

10. A pendulum of length 0.5 meters has a bob of mass 0.1 kg. If the pendulum is released from rest at an angle of 15 degrees, find the total energy in the system.

11. When the amplitude of oscillations of a pendulum is small, then $\sin\theta$ is nearly equal to θ (why?), and the nonlinear equation is approximated by the linear equation $\theta'' + (g/l)\theta = 0$.

a) Show that the approximate linear equation has a solution of the form $\theta(t) = A\cos\omega t$ for some value of ω satisfying the initial conditions $\theta(0) = A$, $\theta'(0) = 0$. What is the period of the oscillation?

b) A 650 lb wrecking ball is suspended on a 20 m cord from the top of a crane. The ball, hanging vertically at rest against the building, is pulled back a small distance and then released. How soon does it strike the building?

12. Consider the nonlinear dynamical system

$$x' = y, \quad y' = -1 - y + x^2.$$

(a) Find the critical points, nullclines, and direction field in the different regions in the plane. **(b)** Given that one of the critical points is an asymptotically stable spiral, determine the type and stability of the other equilibria. **(c)** Draw several key orbits.

13. Consider the nonlinear dissipative system

$$x' = y, \quad y' = -x - y^3.$$

Show that the function $V(x, y) = x^2 + y^2$ decreases along any orbit (i.e., $(d/dt)V(x(t), y(t)) < 0$), and state why this proves that every orbit approaches the origin as $t \to +\infty$.

14. The dynamics of a dissipative mechanical system is given by the differential equation (Newton's law)

$$mx'' = -kx' - V'(x), \quad k > 0.$$

where $V(x)$ is the potential energy in the case $k = 0$. (Recall, V' denotes the x-derivative.) Show that the damping term has the effect of dissipating the energy, $E = \frac{1}{2}m(x')^2 + V(x)$, in the system; precisely, show that

$$\frac{dE}{dt} = -k(x')^2 < 0.$$

15. (Frictional pendulum) Suppose that a simple pendulum is also subject to a frictional force in its joint of magnitude $k\theta'$, where k is the damping constant. **(a)** What is the equation of motion? **(b)** If $E = \frac{1}{2}\omega^2 + (g/l)(1 - \cos\theta)$, show that $E' = -k\omega^2 \le 0$. Interpret this result. **(c)** If $\theta(0) = \pi/12$ and $\theta'(0) = 0$, sketch approximately the orbit in the $\theta\omega$ phase plane.

5.3 Applications

In the last section we saw examples of nonlinear systems in physics. Nonlinear equations also play a central role in population ecology, epidemiology, virology, biochemistry, physiology, and many other areas in the life sciences, as well as in economics and social sciences. In the next few sections we examine some of these models. They are easily understood, without a significant background, using the mathematical tools we now have available.

5.3.1 The Lotka–Volterra Model

We formulate and study a simple model of predator–prey dynamics. Let $x = x(t)$ be the prey population and $y = y(t)$ be the predator population. We can think of rabbits and foxes, food fish and sharks, or any consumer–resource interaction, including herbivores and plants.

If there are no predators we assume the prey dynamics is $x' = rx$, or exponential growth, where r is the positive per capita growth rate. In the absence of prey, we assume that the predator dies via $y' = -my$, where m is the per capita mortality rate. When there are interactions, we must include terms in the dynamics that decrease the prey population and increase the predator population. To determine the form of the predation term, we assume that the rate of predation, or the number of prey consumed per unit of time, per predator, is proportional to the number of prey. That is, the rate of predation, per predator, is ax, where $a > 0$ is a constant. In ecology, this is called the predator's *functional response*. Thus, if there are y predators, then the rate that prey is consumed is axy. This interaction term is proportional to xy, the product of the number of predators and the number of prey. For example, if there were 20 prey and 10 predators, there would be 200 possible interactions; only a fraction of those, a, is assumed to result in a kill. The parameter a, called the *capture efficiency*, depends upon the fraction of encounters and the success of the encounters.

The prey consumed increases the population of predators by the rate εaxy, where ε is the *conversion efficiency*, or yield, of the predator population. (Or, the number of predators produced by consumption of a single prey; one prey consumed does not mean one predator born.[2] Therefore, we obtain the simplest model of predator–prey interaction, called the **Lotka–Volterra model**:

$$x' = rx - axy,$$
$$y' = -my + bxy,$$

where $b = \varepsilon a$.

This model was developed by A. Lotka and V. Volterra in the mid-1920s, and is really the first, and simplest model in ecology showing how populations can cycle. It was one of the first mechanistic models to explain qualitative observations in natural systems.

Remark 5.8

It is important to point out that the term axy, representing the predation rate, is a common interaction term in science. In chemistry, if two molecules $\mathbf{A}$ and $\mathbf{B}$ react to form a product $\mathbf{C}$, or symbolically, $\mathbf{A} + \mathbf{B} \to \mathbf{C}$, then the law of mass action states that the rate of the chemical reaction is proportional to the product of the concentrations of $\mathbf{A}$ and $\mathbf{B}$, or

$$\text{reaction rate } = k[\mathbf{A}][\mathbf{B}].$$

[2] One can argue that instead of numbers, we should be working with biomass of prey and predators.

The constant k is the rate constant. For diseases, the rate of infection transmission is often taken to be aSI, where S is the number of susceptible individuals and I is the number of infected individuals; the constant a is the transmission rate, or the fraction of encounters that lead to infection of a susceptible. □

To analyze the Lotka–Volterra model we factor the right sides of the equations to obtain

$$x' = x(r - ay), \quad y' = y(-m + bx). \tag{5.20}$$

Here, we have $x, y \geq 0$, so we work in the first quadrant. It is simple to locate the critical points, or equilibrium populations. Setting the right sides equal to zero gives two solutions, $x = 0$, $y = 0$ and $x = m/b$, $y = r/a$. Thus, in the phase plane, the points $(0, 0)$ and $(m/b, r/a)$ represent equilibria. The origin represents extinction of both species, and the nonzero equilibrium represents a possible coexistent state.

To determine the type and stability of the equilibria and orbits we plot the nullclines, the curves in the xy plane where the vector field is vertical ($x' = 0$) and curves where the vector field is horizontal ($y' = 0$). The x-nullclines for (5.20), where $x' = 0$, are $x = 0$ and $y = r/a$. The y-nullclines, where $y' = 0$, are $y = 0$ and $x = m/b$. The orbits cross these lines horizontally. Notice that the equilibrium solutions are the intersections of the x- and y-nullclines.

The nullclines partition the plane into four regions where x' and y' have various signs, and therefore we can obtain a direction field indicating directions of the orbits. See Figure 5.11.

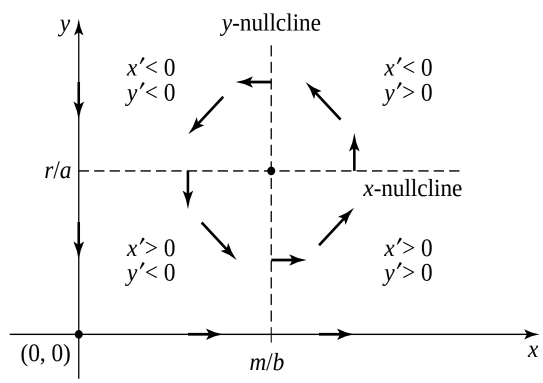

Figure 5.11 Nullclines (dashed) and vector field in regions between the nullclines. The x and y axes are both nullclines and orbits.

Along each nullcline we find the direction of the vector field. For example, on the ray to the right of the equilibrium we have $x > m/b$, $y = r/a$. We know the vector field is vertical so we need only check the sign of y'. We have $y' = y(-m + bx) = (r/a)(-m + bx) > 0$, so the vector field points upward. Similarly we determine the directions along the other three rays. These are shown in the accompanying Figure 5.11. Note that lines $y = 0$ and $x = 0$ are both nullclines and orbits. For example, along $x = 0$ we have $y' = -my$, or $y(t) = Ce^{-mt}$; when there are no prey, the predators die out. Similarly, when $y = 0$ we have $x(t) = Ce^{rt}$, so the prey increase in number.

Finally, we can determine the direction of the vector field in the regions between the nullclines either by selecting an arbitrary point in that region and calculating x' and y', or by just noting the sign of x' and y' in that region. For example, in the northeast quadrant, above and to the right of the nonzero equilibrium, it is easy to see that $x' < 0$ and $y' > 0$; so the vector field points upward and to the left, or in a NW direction. We can complete this task for each region and obtain directions shown in Figure 5.11.

Near $(0, 0)$ the orbits are SE and appear to veer away from the equilibrium. This seems to indicate a saddle structure; clearly, the x and y axes are separatrices. It appears that orbits circle around the nonzero equilibrium in a counterclockwise fashion. But at this time it is not clear if they form closed paths (a center) or spirals (stable or unstable). More analysis is needed and we look at the linearization.

The Jacobian matrix is

$$J(x, y) = \begin{pmatrix} f_x(x, y) & f_y(x, y) \\ g_x(x, y) & g_y(x, y) \end{pmatrix} = \begin{pmatrix} r - ay & -ax \\ by & -m + bx \end{pmatrix}.$$

We have

$$J(0, 0) = \begin{pmatrix} r & 0 \\ 0 & -m \end{pmatrix},$$

which has eigenvalues r and $-m$. Thus $(0, 0)$ is a saddle, confirming our previous observation. For the nonzero equilibrium,

$$J(m/b, r/a) = \begin{pmatrix} 0 & -am/b \\ rb/a & 0 \end{pmatrix}.$$

The characteristic equation is $\lambda^2 + rm = 0$, and therefore the eigenvalues are purely imaginary: $\lambda = \pm\sqrt{rm}\, i$. This is the exceptional case and we cannot conclude that the equilibrium is a center. Again, further analysis is required.

We can obtain the equation of the orbits by the now standard technique of dividing the two equations. We get

$$\frac{y'}{x'} = \frac{dy}{dx} = \frac{y(-m + bx)}{x(r - ay)}.$$

Rearranging and integrating gives

$$\int \frac{r - ay}{y} dy = \int \frac{bx - m}{x} dx + C,$$

or

$$r \ln y - ay = bx - m \ln x + C,$$

which is the algebraic equation for the orbits. It is obscure what these curves are because it is not possible to solve for either of the variables. A different argument is required. If we exponentiate we get

$$y^r e^{-ay} = e^C e^{bx} x^{-m}.$$

Now consider the y nullcline where x *is fixed* at a value m/b, and fix a positive C value (i.e., fix an orbit). The right side of the last equation is a positive number A, and so we can write it as

$$y^r = Ae^{ay}.$$

The left side is a power function and the right side is an exponential. If we plot y^r and Ae^{ay}, we observe that there can be at most two intersections; therefore, this equation can have at most two solutions for y. In summary, along the vertical line $x = m/b$, there can be at most two crossings, and therefore the orbit cannot spiral into or out from the equilibrium point. We conclude that the equilibrium is a center with closed periodic orbits encircling it counterclockwise.

The phase diagram is shown in Figure 5.12. Time series plots of the prey and predator populations are shown in Figure 5.13. When the prey population is high the predators have a high food source and their numbers start to increase, thereby eventually driving down the prey population. Then the prey population gets low, ultimately reducing the number of predators because of lack of food. Then the process repeats, giving cycles.

In summary, the nonzero equilibrium is stable. A small perturbation from equilibrium puts the populations on a periodic orbit that stays near the equilibrium; the system does not return to that equilibrium and the equilibrium is not asymptotically stable. The other equilibrium, the origin, corresponds to extinction of both species and is an unstable saddle point.

5.3.2 Population Ecology

In the Lotka–Volterra model the rate of predation (prey per time, per predator) is assumed to be proportional to the number of prey (i.e., ax). Thinking carefully about this assumption leads to concerns. Increasing the prey density indefinitely leads to an extremely high per predator consumption rate, which

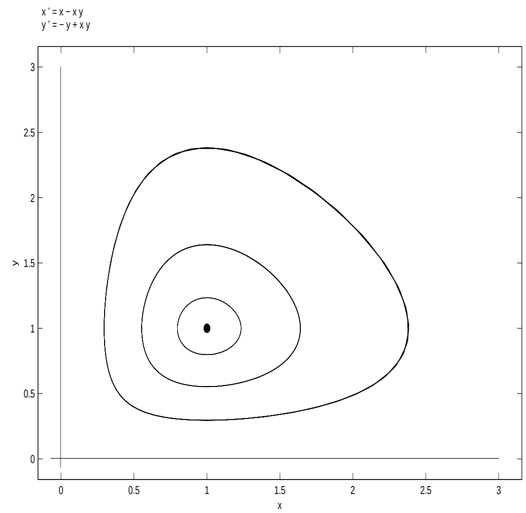

Figure 5.12 Closed, counterclockwise, periodic orbits of the Lotka–Volterra predator–prey model.

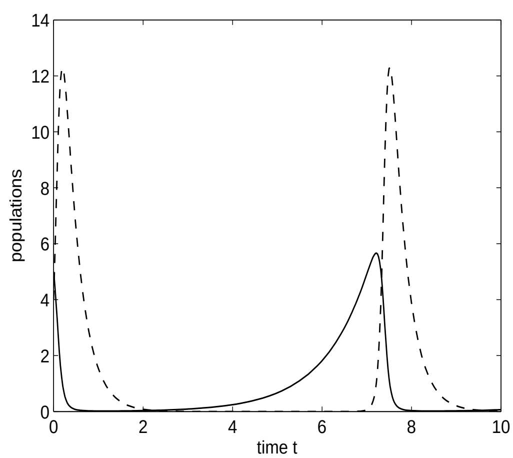

Figure 5.13 Time series solution to the Lotka–Volterra system showing the predator (dashed) and prey (solid) populations.

is clearly impossible for any consumer. It seems more plausible that the rate of predation has a limiting value as the prey density gets large. In the late 1950s, C. Holling developed a functional form that has this limiting property by partitioning the time budget of the predator. He reasoned that the number N of prey captured by a single predator is proportional to the number x of prey and the time T_s allotted for searching.[3] Thus $N = aT_s x$, where the proportionality constant a is the effective encounter rate. But the total time T available to the predator must be partitioned into *search time* T_s and total *handling time* T_h, or $T = T_s + T_h$. The total handling time is proportional to the number captured, $T_h = hN$, where h is the time for a predator to handle a single prey. Hence $N = a(T - hN)x$. Solving for N/T, which is the predation rate, gives

$$\frac{N}{T} = \frac{ax}{1 + ahx}. \qquad \text{(Type II)}$$

This function for the predation rate is called a **Holling type II** response, or the Holling disk equation. Note that $\lim_{x\to\infty} ax/(1 + ahx) = 1/h$, so the rate of predation approaches a constant value. The quantity N/T is measured in prey per time, per predator, so multiplying by the number of predators y gives the predation rate for y predators.

If the encounter rate a is a function of the prey density (e.g., a linear function $a = bx$), the the predation rate is

$$\frac{N}{T} = \frac{bx^2}{1 + bhx^2}, \qquad \text{(Type III)}$$

which is called a **Holling type III** response. Figure 5.14 compares different types of predation rates used by ecologists. For a type III response the predation is turned on once the prey density is high enough; this models, for example, predators that must form a "prey image" before they become aware of the prey, or predators that eat different types of prey. At low densities prey go nearly unnoticed; but once the density reaches an upper threshold the predation rises quickly to its maximum rate.

Replacing the linear per predator feeding rate ax in the Lotka–Volterra model by the *Holling type II response*, we obtain the model

$$x' = rx - \frac{ax}{1 + ahx}y,$$
$$y' = -my + \varepsilon\frac{ax}{1 + ahx}y.$$

[3] We are thinking of x and y as population numbers, but we can also regard them as *population densities*, or animals per area. There is always an underlying fixed area where the dynamics is occurring.

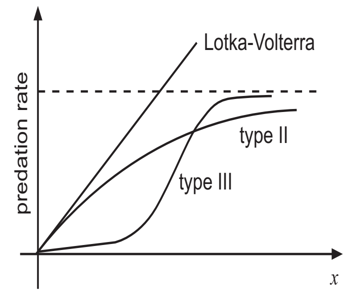

Figure 5.14 Three types of feeding rates, or predation rates, studied in ecology. The predation rate is measured in prey per time, per predator.

We can go another step and replace the linear growth rate of the prey in the model by a more realistic logistic growth term. Then we obtain the **Rosenzweig–MacArthur** model

$$x' = rx\left(1 - \frac{x}{K}\right) - \frac{ax}{1 + ahx}y,$$
$$y = -my + \varepsilon\frac{ax}{1 + ahx}y.$$

Else, a type III response could be used. All of these models have very interesting dynamics. Questions abound. Do they lead to cycles? Are there persistent states where the predator and prey coexist? Does the predator or prey population ever go to extinction? What happens when a parameter, for example, the carrying capacity K, increases? Some aspects of these models are examined in the exercises where the techniques of this chapter come into full play.

Many other types of population models have been developed for interacting species. Competition models, where two species compete for an ecological niche, are among many. A simple **competition model** takes the form

$$x' = xf(x) - axy,$$
$$y' = yg(y) - bxy,$$

where $f(x)$ and $g(x)$ are per capita growth rates for each species. The interaction terms $-axy$ and $-bxy$ are both negative and lead to a decrease in each population. When both interaction terms are positive, then the model is called a **cooperative model**.

Finally, we state a qualification and warning about these types models in biology. In the natural world predator-prey interactions, for example, are very complex and subtle. Interactions are never between only two species, and there are other abiotic effects, such as temperature, that are completely ignored.

Therefore, we should not expect too much from population models. They are not based on physical laws like the laws in physics. Simply put, they are models, or caricatures of natural phenomena. However, models do provide some truths that are useful in understanding these interactions, and their study and development are firmly established in the biology literature.

Dimensionless Equations*

Models are complicated by the presence of several parameters. This makes analysis of models extremely tedious. As well, it is hard to understand the effect of a parameter on the behavior of solutions, even when we perform numerical calculations. Fortunately there is strategy that eliminates some of these complications. Formulating models in terms of **dimensionless variables** often decreases the number of parameters in a problem, leading to a significant economical improvement. This well known practice is standard fare in engineering and biology and belongs to the subject of *dimensional analysis*.

The idea is to define new, dimensionless independent and dependent variables using special combinations of the dimensioned parameters in the problem. The result is a problem that is completely free of dimensions, including the parameters.

Example 5.9

The modified Lotka-Volterra model with logistic growth of the prey is

$$x' = rx\left(1 - \frac{x}{K}\right) - axy,$$
$$y' = -my + bxy,$$

which contains 5 parameters. We show that this model can be reduced down to one with only 2 parameters. First, we find the dimensions of all the parameters and variables in the problem. Clearly, the variables t, x, and y have dimensions *time, prey*, and *predators*. We use the square brackets notation $[\cdot\cdot]$ to denote dimensions; thus

$$[t] = \text{time}, \quad [x] = \text{prey}, \quad [y] = \text{predators}.$$

By comparing terms in the equations, noting that all terms must agree in dimension (apples can't be added to oranges), we can find the dimensions of the parameters.

$$[r] = [m] = \frac{1}{\text{time}}, \quad [K] = \text{prey}, \quad [a] = \frac{1}{\text{time} \cdot \text{predator}}, \quad [b] = \frac{1}{\text{time} \cdot \text{prey}}.$$

We next divide each variable by a combination of parameters that have the same dimension as the variable; then that ratio, which is a new variable, will be dimensionless, meaning dimensions cancel out. There may be several ways to do this. Here, we define new dimensionless variables by

$$\tau = \frac{t}{r^{-1}}, \quad X = \frac{x}{K}, \quad Y = \frac{y}{r/a}.$$

The denominators in expressions above are often called *scales*. We say, for example, the carrying capacity K is a scale for the prey population. Notice, for example, the dimensionless prey population it just the prey population relative to the carrying capacity.

Finally, we change variables in the equations. Noting that derivatives change according to

$$x' = \frac{dx}{dt} = \frac{d(KX)}{d(r^{-1}d\tau)} = Kr\frac{dX}{d\tau}, \quad y' = \frac{dy}{dt} = \frac{d((r/a)dY)}{d(r^{-1}d\tau)} = \frac{r^2}{a}\frac{dX}{d\tau}.$$

Substituting all the new variables into the original equations gives

$$\begin{aligned} Kr\frac{dX}{d\tau} &= rKX(1-X) - a(KX)\frac{r}{a}Y, \\ \frac{r^2}{a}\frac{dX}{d\tau} &= -m\frac{r}{a}Y + bKX\frac{r}{a}Y. \end{aligned}$$

Simplifying gives

$$\begin{aligned} \frac{dX}{d\tau} &= X(1-X) - XY, \\ \frac{dX}{d\tau} &= -\frac{m}{r}Y + \frac{bK}{r}XY. \end{aligned}$$

Therefore, we can define new parameters by

$$\sigma = \frac{m}{r}, \quad \rho = \frac{bK}{r},$$

which are *dimensionless*. Thus, the final dimensionless form of the model is

$$\begin{aligned} \frac{dX}{d\tau} &= X(1-X) - XY, \\ \frac{dX}{d\tau} &= -\sigma Y + \rho XY, \end{aligned}$$

which has only 2 parameters! This problem much easier to analyze. □

The process of non-dimensionalization is extensive and can be very subtle in myriad ways. This important topic of dimensional analysis is discussed, for example, in detail in the classic text by Lin & Segel (1974), or in Logan (2013).

EXERCISES

1. If, in the Lotka–Volterra model, we include a constant harvesting rate h of the prey, the model equations become

$$x' = rx - axy - h$$
$$y' = -my + bxy.$$

Explain how the equilibrium is shifted from that in the Lotka–Volterra model. How does the equilibrium shift if both prey and predator are harvested at the same rate? Compare the observed differences.

2. Non-dimensionalize the Lotka–Volterra model with harvesting in the previous exercise to obtain the system

$$\frac{dX}{d\tau} = X - XY - \eta, \quad \frac{dY}{d\tau} = -\mu X + XY,$$

and show that the two parameters μ and η are dimensionless. Hint: Take

$$\tau = \frac{t}{r^{-1}}, \quad X = \frac{x}{r/b}, \quad Y = \frac{y}{r/a}.$$

3. Modify the Lotka–Volterra model to include *refuge*. That is, assume that the environment always provides a constant number of hiding places where the prey can avoid predators. Argue that

$$x' = rx - a(x-k)y$$
$$y' = -my + b(x-k)y.$$

How does refuge affect the equilibrium populations compared to no refuge?

4. Consider a predator–prey model based on the Lotka–Volterra model, but with predator migration out of the region at a constant rate M:

$$x- = ax - bxy, \quad y' = -cy + dxy - M, \quad c < d.$$

Find the equilibrium and sketch the nullclines. Determine the nature and stability of the equilibrium, and state how the system evolves in time.

5. A simple cooperative model where two species depend upon mutual cooperation for their survival is

$$x' = -kx + axy$$
$$y' = -my + bxy.$$

Find the equilibria and identify, insofar as possible, the region in the phase plane where, if the initial populations lie in that region, then both species become extinct. Can the populations ever coexist in a nonzero equilibrium?

6. In the first quadrant thoroughly analyze the competing species equations:

$$\frac{dx}{dt} = 14x - 2x^2 - xy, \quad \frac{dy}{dt} = 16y - 2y^2 - xy.$$

7. In the first quadrant thoroughly analyze the competing species equations:

$$\frac{dx}{dt} = 14x - \frac{1}{2}x^2 - xy, \quad \frac{dy}{dt} = 16y - \frac{1}{2}y^2 - xy.$$

8. Two populations X and Y grow logistically and both compete for the same resource. A competition model is given by

$$\frac{dX}{d\tau} = r_1 X\left(1 - \frac{X}{K_1}\right) - b_1 XY, \quad \frac{dY}{d\tau} = r_2 Y\left(1 - \frac{Y}{K_2}\right) - b_2 XY.$$

The competition terms are $b_1 XY$ and $b_2 XY$.

a) Nondimensionalize this model by choosing dimensionless variables

$$t = \frac{\tau}{r_1^{-1}}, \quad x = \frac{X}{K_1}, \quad y = \frac{Y}{K_2},$$

thus deriving the dimensionless model

$$x' = x(1-x) - axy, \quad y' = cy(1-y) - bxy,$$

where a, b, and c are appropriately defined dimensionless parameters. Give a biological interpretation of these parameters.

b) In the case $a > 1$ and $c > b$ determine the equilibria, the nullclines, and the direction of the vector field on and in between the nullclines.

c) Determine the stability of the equilibria by sketching a generic phase diagram. How will an initial state evolve in time?

d) Analyze the population dynamics in the case $a > 1$ and $c < b$.

9. For x, $y \geq 0$, consider the system

$$x' = \frac{axy}{1+y} - x, \quad y' = -\frac{axy}{1+y} - y + b,$$

where a and b are positive parameters with $a > 1$ and $b > 1/(a-1)$.

a) Find the equilibrium solutions, plot the nullclines, and find the directions of the vector field along the nullclines.

b) Find the direction field in the first quadrant in the regions bounded by the nullclines. Can you determine from this information the stability of any equilibria?

10. Consider the model

$$x' = y - x$$
$$y' = -y + \frac{5x^2}{4 + x^2},$$

In the first quadrant only, find all equilibrium solutions, sketch nullclines, and indicate the direction of the vector field in all the regions of the first quadrant.

11. *Savannas* are ecosystems characterized by co-dominance of trees and grasses. A simple model for the competition between these two species is given by

$$\frac{dT}{dt} = c_1 T(1 - T) - d_1 T, \quad \frac{dG}{dt} = c_2 G(1 - G) - c_1 T - d_2 G,$$

where T and G are the fractions of trees and grasses, respectively, in the savanna; thus, $T + C = 1$. We assume the constants, all positive, satisfy

$$d_1 < c_1, \quad 1 - \frac{d_1}{c_1} < \frac{c_2 - d_2}{c_1 + c_2}.$$

a) Write a short paragraph explaining the terms in the model.

b) Determine the type and stability of the critical points, and draw the nullclines and key orbits in the phase plane. Is there a coexistent state?

c) Assume the relation among the constants is changed to

$$d_1 < c_1, \quad 1 - \frac{d_1}{c_1} > \frac{c_2 - d_2}{c_1 + c_2}.$$

Explain what happens in this case. Hint: Analyze the phase plane. See F. Accatino et al, *J. Theor. Biol.* (2010), pp 235-242, for a discussion of the effects of rainfall and fire.

12. The dynamics of two competing species is governed by the system

$$x' = x(10 - x - y),$$
$$y' = y(30 - 2x - y).$$

Find the equilibria and sketch the nullclines. Use the Jacobian matrix to determine the type and stability of each equilibrium point and sketch the phase diagram.

13. The dynamics of two competing species is given by

$$x' = 4x(1 - x/4) - xy,$$
$$y' = 2y(1 - ay/2) - bxy.$$

For which values of a and b can the two species coexist? Physically, what do the parameters a and b represent?

14. A predator–prey model is given by

$$x' = rx\left(1 - \frac{x}{K}\right) - f(x)y,$$
$$y' = -my + cf(x)y,$$

where r, m, c, and K are positive parameters, and the predation rate $f(x)$ satisfies $f(0) = 0$, $f'(x) > 0$, and $f(x) \to M$ as $x \to \infty$.

a) Show that $(0, 0)$ and $(K, 0)$ are equilibria.

b) Classify the $(0, 0)$ equilibrium. Find conditions that guarantee $(K, 0)$ is unstable and state what type of unstable point it is.

c) Under what conditions will there be a nonzero equilibrium in the first quadrant?

5.3.3 Epidemics; Chemical kinetics

Measles, flu, chicken pox, whooping cough, ebola, HIV-AIDS, malaria, MRSA, hepatitis—a very short list of the many, many infections that can affect the human species. Some epidemics, like bubonic plague, killed perhaps a third of the inhabitants in Europe in the middle ages. The Spanish flu killed as many as 40-50 million in the early 1900s. It is not surprising that great efforts have made to understand the dynamics of diseases, both from a population scale and now from a microscopic scale. In this section we introduce some simple models of the spread of an infectious disease.

We consider a simple epidemic model where, in a fixed population of size N, the function $I = I(t)$ represents the number of individuals that are infected with a contagious illness and $S = S(t)$ represents the number of individuals that are susceptible to the illness, but not yet infected. We also introduce a removed class where $R = R(t)$ is the number who cannot get the illness because they have recovered permanently, are naturally immune, or have died. We assume

$$N = S(t) + I(t) + R(t),$$

and each individual belongs to only one of the three classes. Observe that N includes the number who may have died. The evolution of the illness in the population can be described as follows. Infectives communicate the disease to susceptibles with a known infection rate; the susceptibles become infectives who have the disease a short time, recover (or die), and enter the removed class. Our goal is to set up a model that describes how the disease progresses with time. These models are called **SIR models**.

In this model we make several assumptions. First, we work in a time frame where we can ignore births and immigration. Next, we assume that the population mixes homogeneously, where all members of the population interact with one another to the same degree and each has the same risk of exposure to the disease. Think of measles, the flu, or chicken pox at an elementary school or small college campus. We assume that individuals get over the disease quickly, so we are not modeling tuberculosis, AIDS, or other long-lasting or permanent diseases. Of course, more complicated models can be developed to account for all sorts of these factors, and other factors such as vaccination, the possibility of reinfection, and so on.

The disease spreads when a susceptible comes in contact with an infective. A reasonable measure of the number of contacts between susceptibles and infectives is $S(t)I(t)$. For example, if there are five infectives and twenty susceptibles, then one hundred contacts are possible. However, not every contact results in an infection. We use the letter a to denote the **effective transmission coefficient**, or the fraction of those contacts that usually result in infection. For example, a could be 0.02, or 2 percent. The parameter a is the product of two effects, the fraction of the total possible number of encounters that occur, and the fraction of those that results in infection. The constant a has dimensions time^{-1} per individual. The quantity $aS(t)I(t)$ is the infection rate, or the rate that members of the susceptible class become infected. Observe that this model is the same as the law of mass action in chemistry where the rate of chemical reaction between two reactants is proportional to the product of their concentrations; it is also the same as the Lotka–Volterra interaction model. Therefore, if *no other processes* are included, we would have

$$S' = -aSI, \quad I' = aSI.$$

But, as individuals get over the disease, they become part of the removed class R. The **recovery rate** r is the fraction of the infected class that ceases to be infected; thus, the rate of removal is $rI(t)$. The parameter r is measured in time^{-1} and $1/r$ can be interpreted as the average time to recover. Therefore, we modify the last set of equations to get

$$S' = -aSI, \tag{5.21}$$

$$I' = aSI - rI. \tag{5.22}$$

These are our working equations. We do not need an equation for R' because R can be determined directly from $R = N - S - I$. At time $t = 0$ we assume there are I_0 infectives and S_0 susceptibles, but no one yet removed. Thus, initial conditions are given by

$$S(0) = S_0, \quad I(0) = I_0, \tag{5.23}$$

and $S_0 + I_0 = N$. SIR models are commonly diagrammed as in Figure 5.15 with S, I, and R compartments and with arrows that indicate the rates that individuals progress from one compartment to the other. An arrow entering a compartment represents a positive rate and an arrow leaving a compartment represents a negative rate.

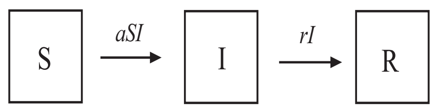

Figure 5.15 In an SIR model: compartments representing the number of susceptibles, the number of infectives, and the number removed, and the flow rates in and out of the compartments.

Qualitative analysis can help us understand how an orbit, or solution curve $S = S(t)$, $I = I(t)$, behaves in the first quadrant of the SI phase plane. Refer to Figure 5.16. First, the initial value must lie on the straight line $I = -S + N$. Where then does the orbit go? Note that S' is always negative so the orbit must always move to the left, decreasing S. Also, because $I' = I(aS - r)$, we see that the number of infectives increases if $S > r/a$, and the number of infectives decreases if $S < r/a$. This information gives us the direction field; left of the I nullcline, or vertical line $S = r/a$, the curves move down and to the left and to the right the curves move up and to the left. Observe that we are assuming $r/a < N$. (The other case is requested in the exercises.)

If the initial condition is at point P in Figure 5.16, the orbit goes directly down and to the left until it intersects $I = 0$, and the disease dies out. If the initial condition is at point Q, then the orbit increases to the left, reaching a maximum at $S = r/a$. Then it decreases to the left and ends on $I = 0$.

There are two questions remaining, namely, how steep the orbit is at the initial point, and where on the S axis does the orbit terminate. Figure 5.16 anticipates the answer to the first question. The total number of infectives and susceptibles cannot go above the line of possible initial conditions, $I + S = N$, and therefore the slope of the orbit at $t = 0$ less than -1, the slope of the line $I + S = N$. To analytically resolve the second issue we can obtain a relationship

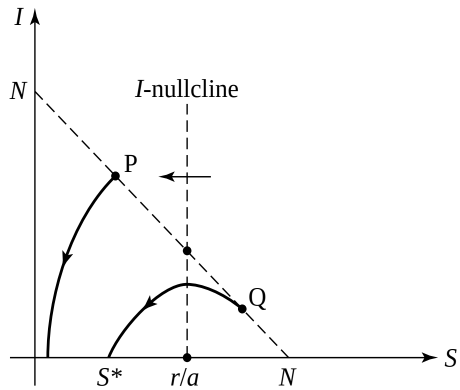

Figure 5.16 The SI phase plane showing two orbits in the case $r/a < N$. One starts at P and and one starts at Q, on the line $I + S = N$. The second shows an epidemic where the number of infectives increases to a maximum value and then decreases to zero; S^* represents the number that does not get the disease.

between S and I along a solution curve as we have done in previous examples.

Dividing the equations (5.21)–(5.22) we obtain

$$\frac{I'}{S'} = \frac{dI/dt}{dS/dt} = \frac{dI}{dS} = \frac{aSI - rI}{-aSI} = -1 + \frac{r}{aS}.$$

Thus

$$\frac{dI}{dS} = -1 + \frac{r}{aS}.$$

Integrating both sides with respect to S (or separating variables) yields

$$I = -S + \frac{r}{a}\ln S + C,$$

where C is an arbitrary constant. From the initial conditions, $C = N - (r/a)\ln S_0$. So the solution curve, or orbit, is

$$I = -S + \frac{r}{a}\ln S + N - \frac{r}{a}\ln S_0 = -S + N + \frac{r}{a}\ln\frac{S}{S_0}.$$

This curve can be graphed with a calculator or computer algebra system, once parameter values are specified. Figure 5.16 shows the orbits. Notice that the orbits cannot intersect the I axis where $S = 0$, so it must intersect the S axis at $I = 0$, or at the root S^* of the nonlinear equation

$$-S + N + \frac{r}{a}\ln\frac{S}{S_0} = 0.$$

See Figure 5.16. This root represents the number of individuals who do not get the disease. Once parameter values are specified, a numerical approximation of S^* can be obtained. In all cases, the disease dies out because of lack of infectives. Therefore, a qualitative analysis of the phase plane gives a good resolution of the disease dynamics.

Generally, we are interested in the question of whether there will be an epidemic when there are initially a small number of infectives. The number

$$R_0 = \frac{aS(0)}{r}$$

is a threshold quantity called the **basic reproductive number**, and it determines if there will be an epidemic. For example, suppose there be a single infective at $t = 0$. That person infects susceptibles at the rate $aIS = a \cdot 1 \cdot S(0)$. Also, that infective has the illness an average time of $1/r$. Therefore a single infective would infect, on the average, $R_0 = aS(0)/r$ individuals. If $R_0 > 1$ there will be an epidemic (the number of infectives increase), and if $R_0 < 1$ then the infection dies out. We think of R_0 as the number of secondary infections produced by a single infective.

In this problem we did not do a critical point analysis. Note that all the critical points are nonisolated, lying along the S axis. This makes our previous results about isolated critical points unusable.

Next we consider a simple model of how an enzyme works to break down a substrate molecule. Enzymes are catalysts that are present in low concentrations but have a rapid effect on a reaction which would otherwise take too much time to be effective in cellular processes. This is a classical problem in biochemistry, and it is the first step in studying other chemically reacting systems, such as activation and inhibition, metabolic pathways, cooperative phenomena, and even virology. The key idea in formulating dynamical equations for the concentrations of chemical constituents is the **Law of Mass Action**, which states that a reaction of the form

$$mX + nY \stackrel{k}{\to} \text{products},$$

where m molecules of X and n molecules of Y react to form products, has rate $r = kx^m y^n$. That is, it is proportional to the products of the reactant concentrations x and y raised to the powers of the number of the molecules involved in the reaction. The proportionality constant k is called the *rate constant*.

Example 5.10

(**Enzyme kinetics**) Among the significant applications of nonlinear systems are the models of chemical kinetics. These describe how concentrations of chemical species evolve in reactions. In this example we give a simple compartmental model of Michaelis–Menten enzyme kinetics, where a substrate S is reduced by an enzyme E. (An example is the hydrolysis of sucrose by invertase.) The basic enzyme reaction is

$$S + E \rightleftarrows C \to P + E, \tag{5.24}$$

where molecules S (substrate) and E (enzyme) combine to form a complex molecule C. The complex C breaks up to form a product molecule P and the original enzyme E, or the reverse reaction, where C disassociates into the original reactants S and E. There are three reactions in (5.24), namely

$$S + E \stackrel{k_1}{\to} C, \qquad C \stackrel{k_{-1}}{\to} S + E, \qquad C \stackrel{k_2}{\to} P + E$$

. The initial formation of the complex C is a rapid reaction, and usually the initial concentration of the enzyme is small compared to the substrate. The only assumption required to reduce the chemical equation to a dynamical model is the *Law of Mass Action*. As remarked above, it states that the rate of a chemical reaction is proportional to the product of the concentrations entering the reaction. The k's in the chemical reactions above are the proportionality constants and they are called the *rate constants*. If we denote the concentrations of S, E, C, and P by lower case letters s, e, c, and p, then the Law of Mass Action states that:

- S and E combine to form C at rate $k_1 se$
- C dissociates into S and E at rate $k_{-1}c$ and into P and E at rate $k_2 c$.

Therefore, the rates of change of the four chemical species are

$$\begin{aligned}
\frac{ds}{dt} &= -k_1 se + k_{-1}c, \\
\frac{de}{dt} &= -k_1 se + (k_{-1} + k_2)c, \\
\frac{dc}{dt} &= k_1 se - (k_{-1} + k_2)c, \\
\frac{dp}{dt} &= k_2 c.
\end{aligned}$$

Initially, we assume $s(0) = s_0$, $e(0) = e_0$, $p(0) = 0$, and $c(0) = 0$, where e_0 is generally very small. Notice that once c is determined, then p is determined by direct integration; and p does not enter the first three equations. Thus, there are only three equations for s, e, and c. Furthermore, we instantly see that $e' + c' = 0$ and $s' + c' + p' = 0$, or $e + c$ and $s + c + p$ are constants for all t. Evaluating these conserved quantities at $t = 0$ gives the relations

$$e + c = e_0, \qquad s + c + p = s_0.$$

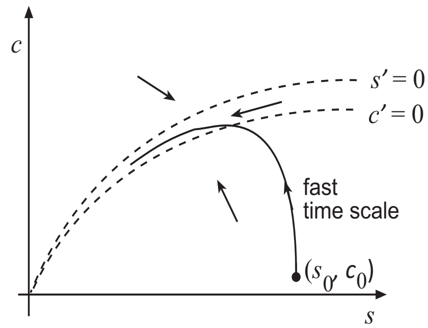

Figure 5.17 The sc phase plane for (5.25)–(5.26) showing the two nullclines (*dashed*) $s' = 0$ and $c' = 0$, as well as the direction field. An orbit beginning at (s_0, c_0) rises steeply and rapidly to cross the $c' = 0$ nullcline; there it is trapped in between the nullclines as t approaches the origin, a stable node. To illustrate the effects, the figure has been expanded vertically; in an actual case, the nullclines lie much closer to the s axis.

We can see that if e_0 is small, then $c(t)$ is small, which makes sense chemically. These equations allow elimination of e and the reduction to a system of *two* nonlinear equations for s and c,

$$\frac{ds}{dt} = -k_1 e_0 s + (k_{-1} + k_1 s)c, \tag{5.25}$$

$$\frac{dc}{dt} = k_1 e_0 s - (k_2 + k_{-1} + k_1 s)c. \tag{5.26}$$

We can easily perform a phase plane analysis to find the structure of the orbits in sc space. Refer to Figure 5.17. The nullclines are

$$c = \frac{k_1 e_0 s}{k_{-1} + k_1 s} \quad (s \text{ nullcline}), \quad c = \frac{k_1 e_0 s}{k_{-1} + k_2 + k_1 s} \quad (c \text{ nullcline})$$

Both are Type 2 (hyperbolic) curves. By comparing denominators, it is clear that the s nullcline lies slightly above the c nullcline. The direction arrows in the three regions are easily found from the differential equations. The orbit increases rapidly in s, changing little, and then becomes trapped between the nullclines as it approaches the origin.

We want to confirm analytically some of our deductions using the assumption that the initial enzyme concentration e_0 is very small in such reactions; this means $c(t)$ is small as well. Let's make a transformation of the independent variable $c(t)$ to a new independent variable $y(t) = c(t)/e_0$. Then the c equation becomes

$$\frac{dy}{dt} = k_1(1 - y)s - (k_{-1} + k_2)y,$$

and the s equation becomes

$$\frac{ds}{dt} = e_0[k_{-1}y - k_1(1-y)s].$$

Comparing the last two equations we observe that s changes very slowly compared to c because of the small factor e_0 on the right side of the s equation. Therefore, we can make the approximation that s is *essentially a constant*, $s(t) \approx s_0$, at early times on the first portion of the steep orbit (Figure 5.17); the formation of the complex is rapid. This type of assumption is called a *quasi-steady state assumption*, and it is common in complex, multiple chemical reactions when one of the reactions is very fast; it equilibrates quickly and can be ignored. Under this assumption the y equation behaves like a first-order autonomous equation, which can be handled by a simple phase line diagram. Its equilibrium is

$$y^* = \frac{k_1 s}{k_{-1} + k_2 + k_1 s}, \qquad s \approx \text{constant},$$

or

$$c^* = \frac{e_0 s}{K + s}, \qquad K = \frac{k_{-1} + k_2}{k_1}.$$

Finally, on this fast regime, the p equation states that the product is produced at rate $k_2 c^*$ or

$$\text{Speed of the reaction} = \frac{k_2 e_0 s}{K + s}.$$

This famous equation is called the *Michaelis–Menten law* in biochemistry. The approximations describe the initial phase of the enzyme reaction, which is on a very fast time scale. After this initial phase, when the orbit becomes trapped between the nullclines, the reaction is on a much slower time scale. There, the orbit is approximately equal to the nullcline. Figure 5.18 shows a generic simulation of time series plots. □

EXERCISES

1. In the SIR model analyze the phase plane diagram in the case $r/a > N$. Does an epidemic occur in this case?

2. Referring to Figure 5.16, qualitatively sketch the shapes of the times series plots $S(t)$, $I(t)$, and $R(t)$ vs, t on the same set of axes when the initial point is Q.

3. In a population of 200 individuals, 20 were initially infected with an influenza virus. After the flu ran its course, it was found that 100 individuals did not contract the flu. If it took about 3 days to recover, what was the transmission coefficient a? What was the average time that it might have taken for someone to get the flu?

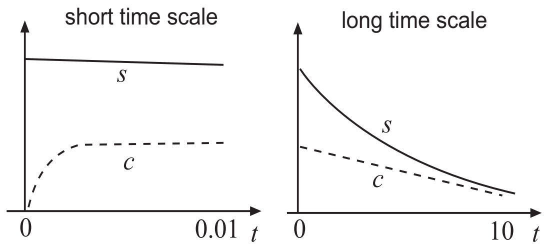

Figure 5.18 Time series plots of $c(t)$ (dashed) and $s(t)$ on two time regimes. The left plot shows the concentrations on a very fast time scale when s is approximately constant and c rapidly rises. The right plot shows the concentrations on a slow time scale, when the orbit is between the nullclines; on this plot the fast time changes are invisible.

4. In a population of 500 people, 25 have a contagious illness. On the average it takes about 2 days to contract the illness and 4 days to recover. How many in the population do not get the illness? What is the maximum number of infectives that occurs?

5. In a constant population, consider an **SIS model** where susceptibles become infected, and then become susceptible immediately after recovery. Assume an infection rate aSI and recovery rate rI. Write the model equations and reformulate the equations as a *single* differential for the infected class. Fully analyze the dynamics of the disease. Can you think of an example of such a disease?

6. Beginning with the SIR model, assume that susceptible individuals are vaccinated at a constant rate ν. Formulate the model equations and describe the progress of the disease if, initially, there are a small number of infectives in a large population.

7. (**SIRS Model**) Beginning with the SIR model, assume that recovered individuals can lose their immunity and become susceptible again after an average recovery period of time μ. That is, the rate recovered individuals become susceptible is μR.

   a) Draw a compartmental diagram.

   b) Formulate a two-dimensional system of equations for S and I.

   c) Find the two equilibria.

   d) By sketching the nullclines and vector field, show that the disease-free equilibrium is unstable. Identify the type of equilibrium?

e) Can you determine whether the nonzero equilibrium (the endemic state) is stable or unstable? What does it appear to be?

8. Analyze an SIR disease model when susceptibles are removed from the population at a per capita rate μ. Thus,

$$\frac{dS}{dt} = -aSI - \mu S,$$
$$\frac{dI}{dt} = aSI - rI,$$
$$\frac{dR}{dt} = rI + \mu S.$$

Proceed as in the SIR model, noting the differences in the dynamics. Hint: Note that the last equation is independent of the first two.

9. (**HIV**) In a simplified model of HIV infection, let x be the population of susceptibles and y the population of those who are infected with HIV, but not AIDS. Suppose there is a constant supply b of susceptibles to the population, and suppose the natural death rate of both susceptibles and infecteds is μ; also, assume that those infected with HIV get AIDS at the rate cy and are thus removed from the HIV population. Finally, assume that the infection rate is
$$ax\frac{y}{x+y},$$
or, the per capita rate of becoming infected is proportional to the fraction of infectious individuals. Draw a compartmental diagram for susceptibles and infectives showing the rates between them and write down the model equations. Show that there is a threshold value a^* of a given by $a^* = \mu + c$ such that the infection dies out for $a < a^*$, and for $a > a^*$ the infection becomes endemic. Verify this conclusion numerically using the values $b = 1000$, $\mu = 0.03$, $c = 0.087$ with $a = 0.1$ and 0.13.

10. For each of the following: **(a)** Find the rates of the reactions and the dynamical equations for the concentrations of the chemical constituents. **(b)** Reduce the equations to the fewest number possible. **(c)** Sketch the phase line or phase plane diagram if appropriate. Note: X, Y, Z, W denote chemical species with concentrations x, y, z, w. Constituents A and B are assumes to be held at constant concentrations a and b.

a) $X \xrightarrow{k} 2Y$.

b) $A + X \xrightarrow{k_1} 2X,\ \ 2X \xrightarrow{k_{-1}} A + X$.

c) $A + X \xrightarrow{k_1} 2X,\ \ 2X \xrightarrow{k_{-1}} A + X,\ \ X + B \xrightarrow{k_2} P$.

d) $2X + Y \xrightarrow{k} 5X + Z$.

e) $X \overset{k_1}{\to} Y + Z,\ \ Y + Z \overset{k_{-1}}{\to} X,\ \ X + W \overset{k_2}{\to} 2X + Y.$

f) $X \overset{k_1}{\to} Y,\ \ Y \overset{k_2}{\to} X + Z.$

11. (Computational) In the Michaelis Menten equations, (5.25)—(5.26), take $k_1 = 10^9$, $k_{-1} = 10^5$, $k_2 = 10^3$, $s_0 = 10^{-3}$, $e_0 = 10^{-5}$, and $c_0 = 0$. Use software to numerically solve the problem over the time interval $0 \leq t \leq 2(10)^{-5}$.

12. If we modify an SIR model by considering births and deaths we obtain the dynamical equations

$$S' = \mu N - aSI - \mu S,\ \ I' = aSI - rI - \mu I,\ \ R' = rI = \mu R.$$

  a) Show that the total population is the constant N. (This is valid because births and deaths are balanced.)

  b) Rewrite the first two equations by making a change of independent variables to $x = S/N$ and $y = aI$. In your equations use the constant $R_0 = aN/(r + \mu)$.

  c) Assume $R_0 > 1$ in your equations in part (b), and investigate the dynamics.

  d) Draw a phase diagram and comment on the results.

13. (**SEI model**) In this epidemic model there is an endemic, or latent, state E between the susceptible and infective states consisting of those who have contracted the disease but are not yet infectious. The model is

$$S' = -aSI, \quad E' = aSI - kE, \quad I' = kE.$$

  a) Draw a compartmental diagram and explain the model.

  b) Reduce the equations to two and draw a phase plane diagram.

  c) How does this model differ in behavior to that of an SI disease studied earlier.

5.3.4 Malaria*

When another organism is involved in the transmission of a disease, that organism is called a vector. These types of diseases are not contagious, or spread by contact, such as in the flu, measles, or some sexually transmitted diseases. In the case of malaria, the mosquito is a vector in the transmission of malaria to different individuals, and the human is also a vector in transmitting the disease among mosquitos; so, this is a criss-cross infection. Malaria affects more individuals worldwide than any other disease, especially in tropical areas. Other vector diseases transmitted by mosquitos include West Nile virus, dengue and yellow fever, and filariasis.

The malaria culprit, from the human point of view, is the female *Anopheles* mosquito. The infectious agent is a protozoan parasite that is injected into the blood stream by a mosquito when she is taking a blood meal, which is necessary for the development of her eggs. The parasite develops inside the host and produces gametocytes which then can be taken up by another biting mosquito.

We have to make some highly simplifying assumptions to obtain a tractable model. We present the classic model of R. Ross, developed in 1911, and modified by G. Macdonald in 1957. Sir Ronald Ross is given credit for first understanding and modeling the complex malarial cycle, a feat for which he was awarded the Nobel Prize.[4]

We assume that human victims have no immune system response and that they eventually recover from the disease without dying. We assume the mosquito and the human populations are approximately constant. Thus, the disease dynamics is fast compared to the dynamics of either hosts or mosquitos Let H_T and M_T be the total number of hosts (humans) and total number of mosquitos, respectively, in a fixed region; both are assumed to be constant. Furthermore, let

$$\begin{aligned} H(t) &= \text{ number of infected hosts (humans)} \\ M(t) &= \text{ number of infected mosquitos} \end{aligned}$$

First we consider the hosts. The rate that a human gets infected depends on the number of mosquitos, the biting rate a (bites per time), and b, the fraction of bites that lead to an infection of a human, and the probability of the mosquito encountering a susceptible human. The fraction of susceptible humans is $(H_T - H)/H_T$. Finally, we assume that the per capita recovery rate of infected humans is r, where $1/r$ is the average time to recovery. Therefore, the rate equation for H is

$$\frac{dH}{dt} = abM\frac{H_T - H}{H_T} - rH,$$

which is the infection rate minus the recovery rate. Notice that the infection rate is proportional to the product of susceptible hosts and infected mosquitos, which should remind the reader of the simple SIR model studied earlier. The

[4] See R. M. Anderson & R. M. May, 1991, *Infectious Diseases of Humans*, Oxford University Press, Oxford UK. This book is the standard reference for diseases, both micro- and macroparasitic.

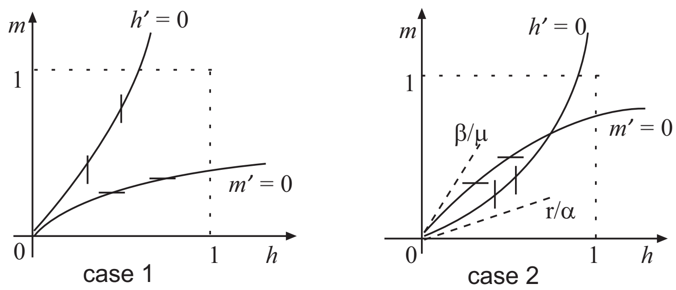

Figure 5.19 The two cases for the malaria model: the nullclines cross only at the origin, and the nullclines cross at the origin and at a nonzero state. The second case will occur only when the slope of the mosquito nullcline exceeds the slope of the host nullcline at the origin, or $\beta/\mu > r/\alpha$.

rate that mosquitos become infected from biting an infected host depends on a and c (the fraction of bites by an uninfected mosquito of an infected human that causes infection in the mosquito). If μ is the per capita death rate of infected mosquitos, then

$$\frac{dM}{dt} = ac(M_T - M)\frac{H}{H_T} - \mu M,$$

and H/H_T is the probability of encountering an infected human. Note that the infection rate is jointly proportional to $M_T - M$, the number of susceptible mosquitos, and the number of infected hosts, again a reminder of an SIR model. Implicit in our assumptions is that birth rates of mosquitos compensates for the death because the total population is constant.

We can immediately simplify these equations by introducing

$$h = \frac{H}{H_T}, \qquad m = \frac{M}{M_T},$$

which are the fractions of the populations that are infected. Then the governing equations become

$$\frac{dh}{dt} = ab\left(\frac{M_T}{H_T}\right)m(1-h) - rh, \tag{5.27}$$

$$\frac{dm}{dt} = ach(1-m) - \mu m. \tag{5.28}$$

For convenience, let's define the parameters

$$\alpha = ab\left(\frac{M_T}{H_T}\right), \qquad \beta = ac.$$

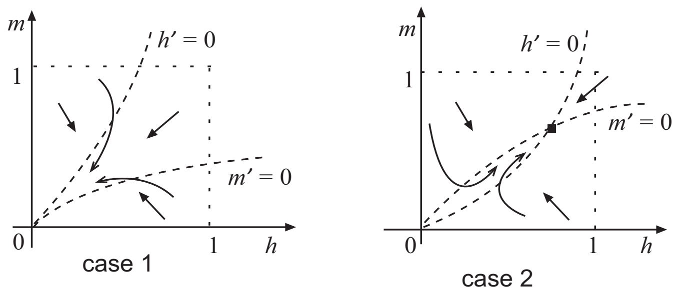

Figure 5.20 Orbits in the two cases. In case 1 the origin is a stable node and the disease epidemic dies out. In case two, the origin is unstable and the disease approaches an endemic state.

Then the equations are

$$\frac{dh}{dt} = \alpha m(1-h) - rh, \tag{5.29}$$

$$\frac{dm}{dt} = \beta h(1-m) - \mu m. \tag{5.30}$$

We can analyze this geometrically in the phase plane in the usual way. Setting the right sides equal to zero gives the nullclines

$$m = \frac{rh}{\alpha(1-h)}, \qquad (h \text{ nullcline}) \tag{5.31}$$

$$m = \frac{\beta h}{\mu + \beta h}, \qquad (m \text{ nullcline}) \tag{5.32}$$

Note that $h = m = 0$ is always an equilibrium. Also, the h nullcline is concave up with a vertical asymptote at $h = 1$; the m nullcline is concave down with a horizontal asymptote at $m = 1$. The two possibilities are shown in Figure 5.19

There will be a nonzero equilibrium only when these curves cross. If there is a nonzero equilibrium, then the slope of the m nullcline must be steeper than the slope of the h nullcline at $h = 0$. Calculating these slopes from (5.31)–(5.32), respectively, we get

$$m'(0) = \frac{r}{\alpha}, \qquad (h \text{ nullcline})$$

$$m'(0) = \frac{\beta}{\mu}. \qquad (m \text{ nullcline})$$

Therefore, for a nonzero equilibrium, we must have the condition

$$\frac{\beta}{\mu} > \frac{r}{\alpha}. \tag{5.33}$$

We show that this nonzero equilibrium is asymptotically stable, which means the infectious populations approach a nonzero endemic state. First, however, let's interpret this result (5.33) in terms of the actual parameter values. We can rewrite (5.33) as

$$\frac{ac}{\mu}\frac{ab\frac{M_T}{H_T}}{r} > 1.$$

The first factor is the rate of infection of mosquitos (ac) times their average lifetime ($1/\mu$). The second factor is the rate of infection of human hosts (abM_T/H_T) times the average length of infection ($1/r$).

We can easily check the stability of the nonzero equilibrium by sketching the direction field. Or, we can approach this analytically by finding the equilibrium and checking the Jacobian matrix. Setting (5.31) equal to (5.32) and solving for h gives

$$h^* = \frac{\alpha\beta - \mu r}{\beta(r+\alpha)}.$$

Then,

$$m^* = \frac{\alpha\beta - \mu r}{\alpha(\mu+\beta)}.$$

Notice that this is a viable equilibrium only if the numerator is positive, which is the same as the condition (5.33). Otherwise it is not viable and the origin, $(0,0)$, is the only equilibrium. The Jacobian matrix at an arbitrary (h,m) is easily

$$J(h,m) = \begin{pmatrix} -\alpha m - r & \alpha(1-h) \\ \beta(1-m) & -\beta h - \mu \end{pmatrix}.$$

Clearly

$$J(0,0) = \begin{pmatrix} -r & \alpha \\ \beta & -\mu \end{pmatrix}.$$

The trace is negative in both cases. The determinant $\mu r - \alpha\beta$ is positive when condition (5.33) holds, and negative when it does not hold. Therefore, the origin (extinction of the disease) is asymptotically stable when it is the only equilibrium, and it is unstable when a nonzero equilibrium exists.

For the nonzero equilibrium

$$J(h^*,m^*) = \begin{pmatrix} -\alpha m^* - r & \alpha(1-h^*) \\ \beta(1-m^*) & -\beta h^* - \mu \end{pmatrix}.$$

The trace is negative and

$$\begin{aligned}\det J(h^*,m^*) &= (\alpha m^* + r)(\beta h^* + \mu) - \alpha\beta(1-m^*)(1-h^*) \\ &= \alpha\beta - \mu r > 0,\end{aligned}$$

Table 5.1 Sample parameters

Parameter	Name	Sample Value
M_T/H_T	population ratio	2
a	biting rate	0.2–0.5 per day
b	effective bites infecting humans	0.5
c	effective bites infecting mosquitos	0.5
r	recovery rate	0.01–0.05 per day
μ	mortality rate	0.05–0.5 per day

by condition (5.33), and after considerable simplification. Thus, (h^*, m^*) is asymptotically stable.

To give some idea of parameter values used in computation, we list a sample set in Table 7.1.

EXERCISES

1. (**Malaria**) In this exercise develop and analyze a simplified version of the malaria model under the condition that r is much less that μ.

   a) Beginning with (5.27)–(5.28), nondimensionalize these equations by rescaling time by taking $\tau = \mu t$. Obtain
   $$\frac{dh}{d\tau} = \lambda m(1-h) - \varepsilon h,$$
   $$\frac{dm}{d\tau} = \eta h(1-m) - m,$$
   where
   $$\varepsilon = \frac{r}{\mu}, \quad \lambda = \frac{ab}{\mu}\frac{M_T}{H_T}, \quad \eta = \frac{ac}{\mu}.$$

   b) Assuming ε is very small, neglect the εh term in the host equation and draw the phase portrait. Include the equilibria, nullclines, direction field, and a local stability analysis for the equilibria.

   c) For the simplified dimensionless model in part (b), with the values given in Table 1, specifically, $a = 0.5$, $r = 0.01$, and $\mu = 0.5$, use a numerical method to draw time series plots of h and m for various initial initial conditions.

2. (**Schistosomiasis***) Schistosomiasis is a macroparasitic disease of humans caused by trematode worms, or blood flukes. Trematodes are a class of flatworms in the phylum *Platyhelminthes*, or helminths. Schistosomiasis

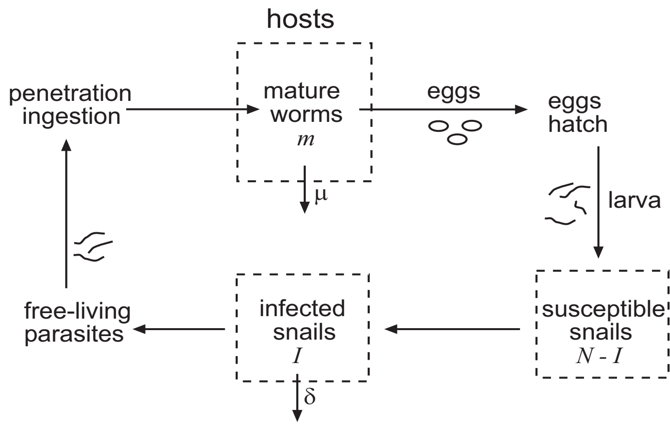

Figure 5.21 Diagrammatic life cycle of schistosome parasites in humans.

is highly prevalent in tropical areas, and it is estimated that hundreds of millions of people suffer from it; it is second only to malaria. The life cycle of the parasite is complicated and involves a definitive host (humans), where maturity and reproduction occur, and a secondary host (e.g., snails), in which the intermediate larval stage develop into infectious larva (cercaria) that are shed and then penetrate, or are ingested, by the definitive host, completing the cycle. Figure 5.21 is a diagrammatic flow chart summarizing the process[5]. Here we consider a simplified model tracking the number of infected snails I and the average worm burden m in the host, which is the total number of mature worms divided by the constant number of hosts. The dynamics for the average number of mature parasites in a host is

$$\frac{dm}{dt} = -\mu m + a\frac{I}{N}, \tag{5.34}$$

where a is the rate that infected snails produce the free living stage larva that infects the host (through ingestion or skin penetration). The factor I/N is the fraction of snails infected, and μ is the per capita mortality rate. The dynamics for the number of infected snails is

$$\frac{dI}{dt} = -\delta I + C(m)(N - I), \quad C(m) = \frac{\alpha m^2}{1 + \beta m^2}. \tag{5.35}$$

Here, δ represents the per capita mortality rate of the infected snails, and $N - I$ is the number of susceptible snails. $C(m)$ is proportional to the rate of production of eggs by (*paired*) female adult worms; the latter includes

[5] See R. M, Anderson & R. M May, *Ibid.*

the rate of hatching of the eggs that eventually produce the infecting, free-living larva. Therefore, C contains several rates in the life cycle and perhaps complicated dependence on the fraction of paired females.

a) Find and draw generic plots the two nullclines in the model.

b) Treat a as a varying parameter, with all the other parameters fixed. Confirm graphically that for small values of a there is only a single equilibrium at $(0, 0)$. Then, as a increases, the slope of the m nullcline decreases until it is tangent to the sigmoid curve and there is a bifurcation at a value $a = a^*$. As a increases further, show there are two nonzero equilibria.

c) Sketch generic phase plane diagrams for the three cases $a < a^*$, $a = a^*$, and $a > a^*$. On your diagrams indicate the nullclines, equilibria, and directions of the vector field.

d) Based upon your plots in (c), find the stability of the equilibria in each case.

e) Draw a bifurcation diagram for equilibrium values of m versus the parameter a. In a few sentences comment on the interpretation of the bifurcation diagram.

f) Find equations that determine the bifurcation point (a^*, m^*).

g) Comment on how a health organization might introduce measures to decrease the value of a.

5.4 Advanced Techniques

5.4.1 Periodic Orbits

We noted a number of times that an exceptional case occurs in the linearization procedure near a critical point: if the associated linearization for the perturbations has a center (purely imaginary eigenvalues) at $(0, 0)$, then the behavior of the nonlinear system at the critical point is undetermined. This suggests that the existence of a periodic solutions (also called *cycles* or *oscillations*), for a nonlinear systems is not easily decided. In this section we discuss special cases when we can be assured that periodic solutions do not exist, and when they do exist. The presence of oscillations in physical and biological systems often represent important phenomena, which is why such solutions are of great interest.

We first state two negative criteria for the nonlinear system

$$x' = f(x, y), \tag{5.36}$$

$$y' = g(x, y), \tag{5.37}$$

where f and g are continuously differentiable functions.

1. (**Equilibrium Criterion**) If the nonlinear system (5.36–5.37) has a cycle, then the region inside the cycle must contain an equilibrium. Therefore, if there are no equilibria in a given region, then the region contains no cycles.
2. (**Dulac's Criterion**) Consider the nonlinear system (5.36–5.37). In a given region of the plane, if there is a function $\beta = \beta(x, y)$ for which

$$\frac{\partial}{\partial x}(\beta f) + \frac{\partial}{\partial y}(\beta g)$$

is of one sign (strictly positive or strictly negative) entirely in the region, then the system cannot have a cycle in that region.

We omit the proof of the equilibrium criterion (it may be found in the references), but we can give the proof of Dulac's criterion because it is a simple application of Green's theorem,[6] encountered in multivariable calculus.

The proof is by contradiction, and it assumes that there *is* a cycle of period p given by $x = x(t)$, $y = y(t)$, $\quad 0 \leq t \leq p$, lying entirely in the region and represented by a simple closed curve C. Assume it encloses a domain R. Without loss of generality suppose that

$$\frac{\partial}{\partial x}(\beta f) + \frac{\partial}{\partial y}(\beta g) > 0.$$

Then, to obtain a contradiction, we make the following calculation using Green's theorem.

$$\begin{aligned} 0 &< \int\int_R \left(\frac{\partial}{\partial x}(\beta f) + \frac{\partial}{\partial y}(\beta g)\right) dA = \int_C (-\beta g dx + bf dy) \\ &= \int_0^p (-\beta g x' dt + bf y' dt) = \int_0^p (-\beta g f dt + bf g dt) = 0, \end{aligned}$$

the contradiction being $0 < 0$. Therefore the assumption of a cycle is false, and there can be no periodic solution. $\square$

[6] Green's theorem: For a nice region R enclosed by a simple closed curve C we have $\int_C P dx + Q dy = \int\int_R (Q_x - P_y) dA$, where C is taken counterclockwise. The functions P and Q are assumed to be continuously differentiable in a open region containing R.

Example 5.11

The system

$$x' = 1 + y^2, \quad y' = x - y + xy$$

does not have any critical points because x' can never equal zero; so this system cannot have cycles. □

Example 5.12

Consider the system

$$x' = x + x^3 - 2y, \quad y' = -3x + y^3.$$

Then

$$\frac{\partial}{\partial x} f + \frac{\partial}{\partial x} g = \frac{\partial}{\partial x}(x + x^3 - 2y) + \frac{\partial}{\partial x}(-3x + y^3) = 1 + 3x^2 + 3y^2 > 0,$$

which is positive for all x and y. Dulac's criterion implies there are no periodic orbits in the entire plane. Note here that $\beta = 1$. □

One must be careful in applying Dulac's criterion. If we find that

$$\frac{\partial}{\partial x}(\beta f) + \frac{\partial}{\partial y}(\beta g) > 0$$

in, say, the first quadrant only, then that means there are no cycles lying entirely in the first quadrant; but there still may be cycles that go out of the first quadrant.

Sometimes cycles can be detected easily in a polar coordinate system. Presence of the expression $x^2 + y^2$ in the system of differential equations often signals that a polar representation might be useful in analyzing the problem.

Example 5.13

Consider the system

$$\begin{aligned} x' &= y + x(1 - x^2 - y^2) \\ y' &= -x + y(1 - x^2 - y^2). \end{aligned}$$

The reader should check, by linearization, that the origin is an unstable spiral point with clockwise rotation. But what happens beyond that? To transform the problem to polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, we note that

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

Taking time derivatives and using the chain rule,

$$rr' = xx' + yy', \quad (\sec^2\theta)\theta' = \frac{xy' - yx'}{x^2}.$$

We can solve for r' and θ' to get

$$r' = x'\cos\theta + y'\sin\theta, \quad \theta' = \frac{y'\cos\theta - x'\sin\theta}{r}.$$

Finally we substitute for x' and y' on the right side from the differential equations to get the polar forms of the equations: $r' = F(r,\theta)$, $\theta' = G(r,\theta)$. Leaving the algebra to the reader, we finally get

$$r' = r(1 - r^2),$$
$$\theta' = -1.$$

By direct integration of the second equation, $\theta = -t + C$, so the angle θ rotates clockwise with constant speed. Notice also that $r = 1$ is a solution to the first equation. Thus we have obtained a periodic solution, a circle of radius $r = 1$, to the system. For $r < 1$ we have $r' > 0$, so r is increasing on orbits, consistent with our remark that the origin is an unstable spiral. For $r > 1$ we have $r' < 0$, so r is decreasing along those orbits. Hence, there is a limit cycle that is approached by orbits from its interior and its exterior. Figure 5.22 shows the phase diagram. □

The Poincaré–Bendixson Theorem

To sum up, through several examples we observed various nonlinear phenomena in the phase plane, including equilibria or critical points, orbits that approach equilibria, orbits that go to infinity, periodic solutions or cycles, and orbits that approach cycles. What have we missed? Is there some other complicated orbital structure that is possible? The famous Poincaré–Bendixson theorem[7] states that the answer is no. The dynamical possibilities for orbits of $x' = f(x,y),\ y' = g(x,y),\ f, g$ continuously differentiable, are limited.

Theorem 5.14

(**Poincaré–Bendixson theorem**) If an orbit is confined to a *closed bounded region* in the plane, then the orbit is a critical point or approaches a critical point as $t \to +\infty$, or the orbit is a cycle or approaches a cycle as $t \to +\infty$. The same result holds true as $t \to -\infty$.

[7] Henri Poincaré (1854–1912) was one of the great contributors to the theory of differential equations and dynamical systems; I. O. Bendixson (1861–1935) was a well known Swedish mathematician.

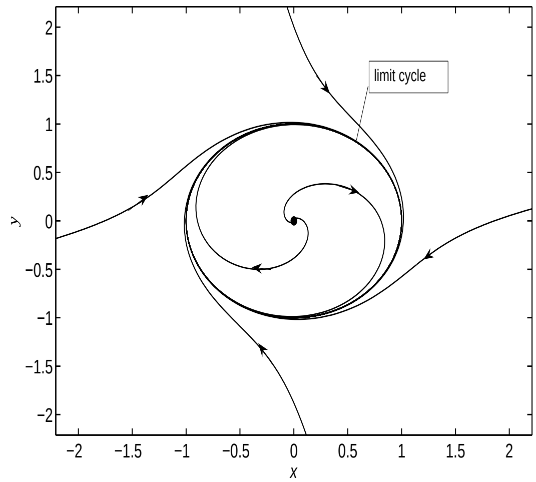

Figure 5.22 Limit cycle. The interior orbits approach the limit cycle clockwise, and the exterior orbits approach it clockwise as well. In this case we say it is an *asymptotically stable* limit cycle.

This theorem is proved in advanced texts. It is important to note that the theorem is *not true* in three dimensions or higher, where orbits for nonlinear systems can exhibit bizarre behavior. For example, orbits in three dimensions can approach sets of *fractal dimension* (strange attractors) showing chaotic behavior. In the next section we will see important applications of this theorem.

EXERCISES

1. Does the system

$$x' = x - y - x\sqrt{x^2 + y^2},$$
$$y' = x + y - y\sqrt{x^2 + y^2},$$

have periodic orbits? Does it have limit cycles?

2. Show that the system

$$x' = 1 + x^2 + y^2,$$
$$y' = (x - 1)^2 + 4,$$

has no periodic solutions.

3. Show that the system

$$x' = x + x^3 - 2y,$$
$$y' = y^5 - 3x,$$

has no periodic solutions.

4. Show that periodic orbits, or cycles, for the dynamical system

$$x' = y, \quad y' = -ky - V'(x)$$

are possible only if $k = 0$.

5. Consider the system

$$x' = x(P - ax + by), \quad y' = y(Q - cy + dx),$$

where $a,\ c > 0$. Show that there cannot be periodic orbits in the first quadrant of the xy plane. Hint: Take $\beta = (xy)^{-1}$.

6. Assume f and g are continuously differentiable everywhere and suppose the nonlinear system $x' = f(x, t)$, $y' = g(x, t)$ has a cycle with a *single* critical point inside. One can prove the critical point cannot be a saddle point. Why do you think this true? Convince yourself by drawing a diagram.

7. Show that the orbits of the system

$$x' = x(x^3 - 2y^3),$$
$$y' = y(2x^3 - y^3),$$

are given by $x^3 + y^3 = 3Cxy$, where C is an arbitrary constant, and sketch several orbits in the phase plane. The curves are called the *folia of Descartes*.

8. In Example 5.13 solve the radial equations for r and θ exactly to find the orbits. Consider each case.

9. A system

$$x' = f(x, y)$$
$$y' = g(x, y),$$

is called a **Hamiltonian system** if there is a function $H(x, y)$ for which $f = H_y$ and $g = -H_x$. The function H is called the *Hamiltonian.* Prove the following facts about Hamiltonian systems.

a) If $f_x + g_y = 0$, then the system is Hamiltonian. (Recall that $f_x + g_y$ is the divergence of the vector field (f, g).)

b) Prove that along any orbit, $H(x, y) = \text{constant}$, and therefore all the orbits are given by $H(x, y) = \text{constant}$.

c) Show that if a Hamiltonian system has an equilibrium, then it is not a source or sink (node or spiral).

d) Show that any conservative dynamical equation $x'' = f(x)$ leads to a Hamiltonian system, and show that the Hamiltonian coincides with the total energy.

e) Find the Hamiltonian for the system $x' = y, \quad y' = x - x^2$, and plot the orbits.

10. In a Hamiltonian system the Hamiltonian given by $H(x, y) = x^2 + 4y^4$. Write down the system and determine the equilibria. Sketch the orbits.

11. A system

$$x' = f(x, y)$$
$$y' = g(x, y),$$

is called a **gradient system** if there is a function $G(x, y)$ for which $f = G_x$ and $g = G_y$.

a) If $f_y - g_x = 0$, prove that the system is a gradient system. (Recall that $f_y - g_x$ is the curl of the two-dimensional vector field (f, g); a zero curl ensures existence of a potential function on nice domains.)

b) Prove that along any orbit, $(d/dt)G(x, t) \geq 0$. Show that periodic orbits are impossible in gradient systems.

c) Show that if a gradient system has an equilibrium, then it is not a center or spiral.

d) Show that the system $x' = 9x^2 - 10xy^2, \quad y' = 2y - 10x^2y$ is a gradient system.

e) Show that the system $x' = \sin y, \ y' = x \cos y$ has no periodic orbits.

12. (*Chemical kinetics*) A chemical reaction, where two molecules of **X** react with a molecule of **Y** to produce the product $\mathbf{X_2Y}$, proceeds in two steps,

$$\mathbf{X} + \mathbf{Y} \to \mathbf{XY},$$
$$\mathbf{X} + \mathbf{XY} \to \mathbf{X_2Y}.$$

By the law of mass action, the rates of these reactions are $r_1 = \alpha xy$ and $r_2 = \beta xz$, where x, y, and z denote the concentrations of **X**, **Y** and **XY**, respectively, and α and β are the rate constants with $2\beta > \alpha$.

a) Show that the concentrations change according to

$$\frac{dx}{dt} = -\alpha xy - \beta xz, \quad \frac{dy}{dt} = -\alpha xy, \quad \frac{dz}{dt} = \alpha xy - \beta xz.$$

b) Show that $x - 2y - z = C$, a constant.

c) Initially, assume $x(0) = 2y(0)$. Show that the system can be reduced to two equations

$$\frac{dx}{dt} = (2\beta - \alpha)xy - \beta x^2, \quad \frac{dy}{dt} = -\alpha xy.$$

d) In the xy phase plane, describe the kinetics of the reaction.

13. **(a)** Describe fully the dynamics of the polar system $r' = r^3 - 3r^2 + 2r$, $\theta' = 1$. **(b)** Write down the system in rectangular coordinates.

14. Sketch the phase plane diagram for each equation. Hint: You might check if the system is Hamiltonian or gradient.

a) $x' = y^2 + 2xy,\ y' = x^2 + 2xy$.

b) $x^2 - 2xy,\ y' = y^2 - x^2$.

c) $x' = x^2 - 2xy,\ y' = y^2 - 2xy$.

d) $x' = \sin^2 x \sin y$,
$y' = -2 \sin x \cos x \cos y$.

5.5 Bifurcations

In Chapter 1 we examined the structure of a single autonomous differential equation $x' = f(x, h)$, where h is a parameter. The equilibria $x = x_e$ satisfy the equation $f(x_e, h) = 0$, so they depend on the parameter h. As h varies, the equilibria and their stability properties vary as well, and we faced the phenomenon of *bifurcation*. In Chapter 4 we saw that two-dimensional linear systems depending on parameters have similar issues, and they have a much richer structure than their one-dimensional counterparts. Here we examine a nonlinear system, which exposes yet another bifurcation issue.

Example 5.15

Consider the model

$$x' = \frac{2}{3}x\left(1 - \frac{x}{4}\right) - \frac{xy}{1+x},$$
$$y' = ry\left(1 - \frac{y}{x}\right), \quad r > 0.$$

In an ecological context, we can think of this system as a predator–prey model. The prey (x) grow logistically and are harvested by the predators (y) with a Holling type II rate. The predator grows logistically, with its carrying capacity depending linearly upon the prey population. The horizontal, y nullclines, are $y = x$ and $y = 0$, and the vertical, or x nullcline, is the parabola $y = \left(\frac{2}{3} - \frac{1}{6}x\right)(x+1)$. The equilibria are $(1,1)$, and $(4,0)$. The system is not defined when $x = 0$ and we classify the y-axis as a line of *singularities*; no orbits can cross this line. The Jacobian matrix is

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} \frac{2}{3} - \frac{1}{3}x - \frac{y}{(1+x)^2} & \frac{-x}{1+x} \\ \frac{ry^2}{x^2} & r - \frac{2ry}{x} \end{pmatrix}.$$

Evaluating at the equilibria yields

$$J(4,0) = \begin{pmatrix} -\frac{2}{3} & -\frac{4}{5} \\ 0 & r \end{pmatrix}, \quad J(1,1) = \begin{pmatrix} \frac{1}{12} & -\frac{1}{2} \\ r & -r \end{pmatrix}.$$

It is clear that $(4,0)$ is a saddle point with eigenvalues r and $-2/3$. At $(1,1)$ we find $\mathrm{tr}J = \frac{1}{12} - r$ and $\det J = \frac{5}{12}r > 0$. Therefore $(1,1)$ is asymptotically stable if $r > \frac{1}{12}$ and unstable if $r < \frac{1}{12}$. So, there is a bifurcation, or change, at $r = \frac{1}{12}$ because the stability of the equilibrium changes. For a large predator growth rate r there is a nonzero persistent state where predator and prey can coexist. As the growth rate of the predator decreases to a critical value, this persistence goes away. What happens then? Let us imagine that the system is in the stable equilibrium state and other factors, possibly environmental, cause the growth rate of the predator to slowly decrease. How will the populations respond once the critical value of r is reached?

Let us carefully examine the case when $r < \frac{1}{12}$. Consider the direction of the vector field on the boundary of the square with corners $(0,0)$, $(4,0)$, $(4,4)$, $(0,4)$. See Figure 5.23. On the left side ($x = 0$) the vector field is undefined, and near that boundary it is nearly vertical; orbits cannot enter or escape along that edge. On the lower side ($y = 0$) the vector field is horizontal ($y' = 0$, $x' > 0$). On the right edge ($x = 4$) we have $x' < 0$ and $y' > 0$, so the vector field points into the square. And, finally, along the upper edge ($y = 4$) we have $x' < 0$ and $y' < 0$, so again the vector field points into the square. The equilibrium at $(1,1)$ is unstable, so orbits go away from equilibrium; but they cannot escape from the square. On the other hand, orbits along the top and right sides are entering the square. What can happen? They cannot crash into each other! (Uniqueness.) So, there must be a counterclockwise limit cycle in the interior of the square (by the Poincaré–Bendixson theorem). The orbits entering the square approach the cycle from the outside, and the orbits coming out of the unstable equilibrium at $(1,1)$ approach the cycle from the inside. Now we can state what happens as the predator growth rate r decreases through the critical value. The persistent

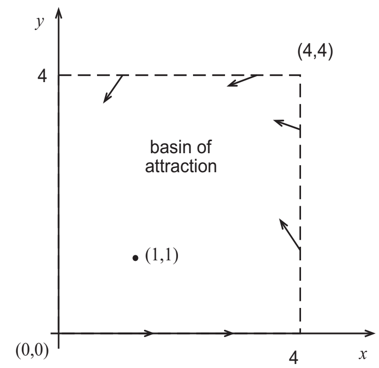

Figure 5.23 A square representing a basin of attraction. Orbits cannot escape the square.

state becomes unstable and a small perturbation, always present, causes the orbit to approach the limit cycle. Thus, we expect the populations to cycle near the limit cycle. A phase diagram is shown in Figure 5.24. □

In this example we used a common technique of constructing a region, called a **basin of attraction**, that contains an unstable spiral (or node), but orbits cannot escape the region. In this case there must be a limit cycle in the region. A similar result holds true for annular type regions (doughnut type regions bounded by concentric simple close curves); if there are no equilibria in an annular region R and the vector field points inward into the region on both the inner and outer concentric boundaries, then there must be a limit cycle in R.

EXERCISES

1. Describe the bifurcations that occur in the system

$$x' = \varepsilon x - y, \quad y' = x + \varepsilon y$$

as ε varies from -0.2 to 0.2. Include plots in your answer.

2. Analyze the dynamics of the system

$$x' = y,$$
$$y' = -x(1-x) + cy$$

for different positive values of c. Draw phase diagrams for each case, illustrating the behavior.

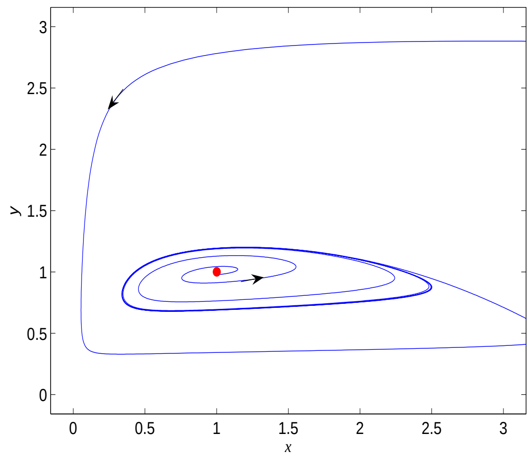

Figure 5.24 Phase diagram showing a counterclockwise limit cycle. Curves approach the limit cycle from the outside and from the inside. The interior equilibrium is an unstable spiral point.

3. A predator-prey population model is governed by the dynamics

$$x' = -x + xy, \quad y' = 4y\left(1 - \frac{y}{K}\right) - 2xy,$$

where K is the carrying capacity of the predator. As K slowly increases from 0.5 to 4.0, determine and describe the bifurcations that occur in the model.

4. Consider the system

$$x' = ax + y - x(x^2 + y^2),$$
$$y' = -x + ay - y(x^2 + y^2),$$

where a is a parameter. **(a)** Show that the system becomes

$$r' = r(a - r^2), \quad \theta' = -1$$

in polar coordinates. **(b)** If $a < 0$ show $(0, 0)$ is a stable spiral. **(c)** If $a > 0$, show that $(0, 0)$ is an unstable spiral. **(d)** Show that there is a limit cycle at $r = \sqrt{a}$.

This is an example of a *Hopf bifurcation*. As the parameter increases from negative to positive, the eigenvalues pass across the imaginary axis in the complex plane, signaling the birth of a limit cycle.

5. Let P denote the carbon biomass of plants in an ecosystem and H the carbon biomass of herbivores. Let ϕ denote the constant rate of primary production of carbon in plants due to photosynthesis. Then a model of plant–herbivore dynamics is given by

$$P' = \phi - aP - bHP,$$
$$H' = \varepsilon bHP - cH,$$

where a, b, c, and ε are positive parameters.

a) Explain the various terms in the model and determine the dimensions of each constant.

b) Find the equilibrium solutions.

c) Analyze the dynamics in two cases, that of high primary production ($\phi > ac/\varepsilon b$) and low primary production ($\phi < ac/\varepsilon b$). Determine what happens to the system if the primary production is slowly increased from a low value to a high value.

6. Consider the system

$$x' = x^2 - h,$$
$$y' = -y.$$

Show that a bifurcation occurs at $h = 0$ because of a change in dynamics. Plot, for $h > 0$, the equilibrium values of x versus h and label the branches as stable or unstable.

7. Consider two competing species where one of the species immigrates or emigrates at constant rate h. The populations are governed by the dynamical equations

$$x' = x(1 - ax) - xy,$$
$$y' = y(b - y) - xy + h,$$

where $a, b > 0$.

a) In the case $h = 0$ (no immigration or emigration) give a complete analysis of the system and indicate in a, b parameter space (i.e., in the ab plane) the different possible behaviors, including where bifurcations occur. Include in your discussion equilibria, stability, and so forth.

b) Repeat part (a) for various fixed values of h, with $h > 0$.

c) Repeat part (a) for various fixed values of h, with $h < 0$.

8. Determine the nature of each equilibrium of the system

$$x' = 4x^2 - a,$$
$$y' = -\frac{y}{4}(x^2 + 4),$$

and show how the equilibria change as the parameter a varies.

9. Give a thorough description, in terms of equilibria, stability, and phase diagram, of the behavior of the system

$$x' = y + (1 - x)(2 - x),$$
$$y' = y - ax^2,$$

as a function of the parameter $a > 0$.

10. Consider the nonlinear system

$$x' = 1 - (a + 1)x + x^2 y,$$
$$y' = ax - x^2 y, \quad a > 0.$$

a) Find the equilibrium and the Jacobian matrix in the first quadrant.

b) Show the equilibrium is stable for $a < 2$ and unstable for $a > 2$.

c) Use a numerical algorithm to detect a limit cycle when $a = 3$.

11. The populations of two competing species x and y are modeled by the system

$$x' = (K - x)x - xy,$$
$$y' = (1 - 2y)y - xy,$$

where K is a positive constant. In terms of K, find the equilibria. Explain how the equilibria change, as to type and stability, as the parameter K increases through the interval $0 < K \le 1$, and describe how the phase diagram evolves. Especially describe the nature of the change at $K = \frac{1}{2}$.

12. (**van der Pol's equation**) In an RCL circuit, replace Ohm's law, $V = RI$, by a nonlinear function $V = f(I)$ of current.

a) Show that the circuit equation becomes $LI'' + f'(I)I' + \frac{1}{C}I = 0$.

b) Specifically, show that if $f(I) = bI^3 - aI$, then

$$LI'' + (3bI^2 - a)I' + \frac{1}{C}I = 0.$$

c) Define new independent and dependent variables defined by $\tau = \sqrt{LC}t$, $x = I/p$, where $p = \sqrt{a/(b)}$, and show that the previous equation can be written

$$x'' + \mu(x^2 - 1)x' + x = 0, \qquad \mu = a\sqrt{\frac{C}{L}}.$$

This is the famous van der Pol equation for a RCL circuit with a active resistance.

d) Reduce the van der Pol equation to a system and show if $0 < \mu < 2$ the origin is an unstable spiral, and if $\mu > 2$ it is an unstable node.

e) Take the case $\mu = 1$ and use software to numerically show that the van der Pol equation has a limit cycle. Hint: Use the initial conditions: $x(0) = 1$, $y(0) = -2$; $x(0) = 2$, $y(0) = 0$; $x(0) = 0.25$, $y(0) = 0$.

13. (*Chaos*) Chaos is a phenomenon where small changes in the initial conditions or parameters in the problem cause dramatic, unpredictable changes in the solution. It is not randomness, but a type of instability in a deterministic system. Consider a special case of the *Duffing equation*

$$x'' + x' - x + x^3 = F\cos t, \;\; x(0) = 1, \;\; x'(0) = 0, \;\; (F = 0.8),$$

which governs a damped, nonlinear spring forced by a periodic source. **(a)** Formulate the equation as a system and use software to sketch the orbit in the phase plane for $0 \le t \le 500$. **(b)** Repeat the same calculation for $F = 0.75$. Observe the dramatic change in the solution, even though the values of F are close. For $F = 0.8$ the orbits wanders seemingly randomly around the phase plane, exhibiting the phenomenon of *chaos*. **(c)** Repeat the calculation for $F = 0.8$ with initial conditions $x(0) = 1.1$, $x'(0) = 0$. Observe a similar dramatic change.

14. (*Macroparasites*) In Exercise 2 of Section 5.3 we examined a helminth parasite infection by tracking the average worm burden in a host and the number of infected secondary hosts (snails). Now consider a different model of a helminth infection that tracks mature worms in the host and the larval population. Let L be the number of larval parasites in the environment and M the number of mature parasites in the hosts. The equations are

$$\frac{dL}{dt} = bM - \lambda LN - \nu L, \quad \frac{dM}{dt} = \lambda LN - \mu M,$$

where N is the total number of hosts, b is the larval per capita birth rate at which adults produce larva, ν is the per capita larval death rate, μ is the per capita parasite death rate, and λ is the force of infection. Interpret this model and give a standard phase plane analysis, sketching the phase diagram in the case that an epidemic breaks out.

15. (*Chemical kinetics*) The system

$$x' = 1 - (a+1)x + \frac{1}{4}x^2y, \quad y' = ax - \frac{1}{4}x^2y, \quad a > 0,$$

represents the kinetics of a reaction where the chemical species undergo oscillations, with $x, y \geq 0$.

a) Show that the only critical point is $(1, 4a)$ and find the eigenvalues of the linearization at that point.

b) Classify the type and stability of the critical point as it depends on a.

c) At what critical value of a is there a bifurcation?

d) Plot the orbits for values of a slightly above and below the critical value.

e) Indicate through calculations that a limit cycle is born as a passes through the critical value. This is another example of a Hopf bifurcation. See Exercise 4.

16. Consider the nonlinear system $x' = x^2 + y$, $y' = x - y + h$, where h is a real parameter. **(a)** Find all the critical points. **(b)** Compute the Jacobian matrix at each critical point. **(c)** Describe the bifurcation(s) in the system.

17. Consider the system

$$x' = y, \quad y' = -cy - x(1-x), c > 2.$$

By analyzing the phase plane diagram for $x \geq 0$, show that there must be an orbit connecting the two critical points $(1, 0)$ and $(0, 0)$ as t goes from $-\infty$ to $+\infty$.

18. Use the Poincaré–Bendixson theorem to show that the system

$$x' = x - y - x^3, \quad y' = x + y - y^3,$$

has a periodic solution. Hint: Show the system has a square as the basin of attraction.

19. For what value of the parameter α does the following system undergo a Hopf bifurcation?

$$x' = x^2(1-x) - xy, \quad y' = xy - \frac{y}{\alpha}.$$

6
Computation of Solutions

The fact is that most differential equations cannot be solved with simple analytic formulas. Therefore we are interested in developing methods to approximate solutions. An approximation can be a formula, or it can arise as a data set or plot obtained by a computer algorithm. The latter forms the basis of modern scientific computation, and it may be the most useful topic in this book for future scientists and engineers.

In this chapter is a very brief account of three methods: the Euler method, the modified Euler method, and the fourth-order Runge–Kutta method. There is a concise discussion of the errors encountered in using these methods. The exercises require access to a computer algebra system or an advanced scientific calculator.

6.1 Iteration*

In this optional first section we introduce the notion of recursion formulas. We review Newton's method and fixed point iteration for finding roots of algebraic equations, and then we expand the idea to differential equations and introduce an iterative procedure called the method of successive approximations, or *Picard's method* (E. Picard, 1856–1941), that leads to a recursive analytic formula that, when applied over and over, gives an approximate formula for the solution to an initial value problem.

From calculus we learn that **Newton's method** is an iterative procedure that approximates a root of the algebraic equation $f(t) = 0$. Newton's method is defined by the recursive equation

$$t_{k+1} = t_k - \frac{f(t_k)}{f'(t_k)}, \quad k = 0, 1, 2, \ldots,$$

where t_0 is a prescribed, first approximation to a root. If t_0 is sufficiently close to the actual root and other conditions, which we do not discuss, hold, then the successive approximations t_1, t_2, $t_3, \ldots$, computed by the recursion formula converge rapidly to the root. It is clear from the formula that the method has difficulties near points where $f'(t) = 0$.

Picard's method is adapted from another classical iterative method, called the **fixed point method**. It is used to approximate solutions of nonlinear algebraic equations in the form $t = g(t)$.

Example 6.1

(**Fixed Point Iteration**) Consider the problem of solving the nonlinear algebraic equation

$$t = \cos t.$$

Graphically, it is clear that is a unique solution because the curves $y = t$ and $y = \cos t$ cross at a single point between 0 and $\pi/2$. Analytically we can approximate the root by making an initial guess t_0 near the root and then successively calculate better approximations via

$$t_{k+1} = \cos t_k \quad \text{for} \quad k = 0, 1, 2, ...$$

For example, if we choose $t_0 = 0.9$, then $t_1 = \cos x_0 = \cos(0.9) = 0.622$, $t_2 = \cos t_1 = \cos(0.622) = 0.813$, $t_3 = \cos t_2 = \cos(0.813) = 0.687$, $t_4 = \cos t_3 = \cos(0.687) = 0.773$, $t_5 = \cos t_4 = \cos(0.773) = 0.716, ...$ Thus we have generated a sequence of approximations 0.9, 0.622, 0.813, 0.687, 0.773, $0.716, ...$ If we continue the process, the sequence converges to some t^*, which is the solution to $t = \cos t$. To three decimal places, $t^* = 0.739$. □

The method of fixed point iteration can be applied to general algebraic equations of the form

$$t = g(t).$$

The iterative procedure is

$$t_{k+1} = g(t_k), \quad k = 0, 1, 2, ...$$

and it will converge to a root t^* provided $|g'(t^*)| < 1$ and the initial guess t_0 is sufficiently close to t^*. The conditions stipulate that the graph of g cannot be too steep (its absolute slope at the root must be bounded by one), and the initial guess must be sufficiently close to the root.

This fixed point iteration idea for algebraic equations can be expanded to find approximate solutions to first-order differential equations, and systems of first-order equations. Consider the initial value problem

$$\text{(IVP)} \qquad \begin{cases} x' = f(t,x), \\ x(t_0) = x_0. \end{cases}$$

The first step is to turn this problem into an equivalent integral equation by integrating the differential equation from t_0 to t and using the fundamental theorem of calculus. Easily this gives the equivalent problem

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds. \tag{6.1}$$

Now we define **Picard iteration**, a type of fixed point iteration. It is based on the integral equation formulation (6.1). The iterative scheme is

$$x_{k+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_k(s))ds, \quad k = 0, 1, 2, ..., \tag{6.2}$$

where where we start the procedure with $x_0(t) = x_0$, a prescribed initial approximation. Then we generate a sequence $x_1(t)$, $x_2(t)$, $x_3(t), \ldots$ of iterates, called *Picard iterates*, that under certain conditions converge to the solution of the original initial value problem, or equivalently, to (6.1).

Example 6.2

Consider the linear initial value problem

$$x' = 2t(1 + x), \quad x(0) = 0.$$

Then the iteration scheme is

$$x_{k+1}(t) = \int_0^t 2s(1 + x_k(s))ds, \quad k = 0, 1, 2, \ldots,$$

Take $x_0(t) = 0$, the initial condition; then

$$x_1(t) = \int_0^t 2s(1 + 0)ds = t^2.$$

Then

$$x_2(t) = \int_0^t 2s(1 + x_1(s))ds = \int_0^t 2s(1 + s^2)ds = t^2 + \frac{1}{2}t^4.$$

Next,

$$x_3(t) = \int_0^t 2s(1 + x_2(s))ds = \int_0^t 2s(1 + s^2 + \frac{1}{2}s^4)ds = t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6.$$

In this manner we generate a sequence of approximations of the solution to the IVP. In this case one can verify that the analytic solution to the IVP is

$$x(t) = e^{t^2} - 1.$$

The Taylor series expansion of this exact solution is

$$x(t) = e^{t^2} - 1 = t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \cdots + \frac{1}{n!}t^{2n} + \cdots,$$

and it converges for all t. Therefore the successive approximations generated by Picard iteration are the partial sums of this power series, and they converge to the exact solution. □

The Picard procedure is especially important from a theoretical viewpoint. The method forms the basis of an existence proof for the solution to a general initial value problem; the idea is to show that there is a limit to the sequence of approximations, and that limit is the solution to the initial value problem. This topic is discussed in advanced texts on differential equations (e.g., see Waltman 1986, or Hirsch et al, 2004). Practically, however, Picard iteration is not useful for problems in science and engineering. There are other methods, based upon numerical algorithms and perturbation methods, that give highly accurate approximations with much less work. We introduce numerical methods in the next section.

Finally, we point out that Picard iteration is guaranteed to converge if the right side of the differential equation $f(t, x)$ is regular enough; specifically, the first partial derivatives of f must be continuous in an open rectangle of the tx plane containing the initial point. However, convergence is only guaranteed locally, in a small interval about t_0.

EXERCISES

1. Consider the initial value problem

$$x' = 1 + x^2, \quad x(0) = 0.$$

Apply Picard iteration with $x_0(t) = 0$ and compute four terms. If the process continues, to what function will the resulting series converge?

2. Apply Picard iteration to the initial value problem

$$x' = t - x, \quad x(0) = 1,$$

to obtain three Picard iterates, taking $x_0(t) = 1$. Plot each iterate and the exact solution on the same set of axes.

3. Consider the initial value problem $x' = -tx$, $x(1) = 2$.

   a) Derive the exact solution $x(t) = 2e^{(1-x^2)/2}$.

   b) Show that the integral equation formulation of the problem is

   $$x(t) = 2 - \int_1^t sx(s)dx.$$

   c) Compute the first three Picard iterates, $x_1(t), \ldots, x_3(t)$.

   d) Plot the exact solution $x(t)$ and $x_3(t)$ on the interval $1 \leq t \leq 2$.

6.2 Numerical Methods

We emphasize again that most differential equations cannot be solved analytically by a simple formula. In this section we develop a class of numerical methods that solve an initial value problem approximately using a computer algorithm. In industry and science, differential equations are almost always solved numerically because most real-world problems lead to models too complicated to solve analytically. And, in the rare case the problem can be solved analytically, the solution is often in the form of a complicated integral or infinite series which must be resolved by a computer calculation anyway. We can never underestimate the importance and utility of numerical methods in science and technology. Many would say that the material in this brief section is the most important in the entire book.

Any computer algebra system can easily implement the numerical algorithms. We can either write a code (a program), or we can use the existing templates or packages given in the software. Students can access these in MATLAB, Maple, Mathematica, R, and many other computer algebra systems. We do not use a specific system in this text, but rather only indicate the algorithms. Specific instructions to access MATLAB topics relevant to this text are found in the Preface.

6.2.1 The Euler Method

We develop numerical approximations using a method called **finite difference methods**. Here is the basic idea. Suppose we want to solve the following initial value problem on the interval $t_0 \le t \le T$:

$$x' = f(t,x), \quad x(t_0) = x_0. \tag{6.3}$$

Rather than seek a continuous solution defined at each time t, we develop a strategy of discretizing the problem to determine numerical sequence of approximations X_0, X_1, X_2, $X_3, \ldots$ at discrete times $t_0 < t_1 < t_2 < \cdots < t_n < \cdots$ in the interval of interest. Therefore, the plan is to replace the continuous-time model (6.3) with an approximate discrete-time model that is amenable to computer solution.

To this end, we divide the interval $t_0 \le t \le T$ into N small segments of constant length h, called the **step size**. Thus the stepsize is

$$h = \frac{T - t_0}{N}.$$

This defines the set of equally spaced discrete times $t_0, t_1, t_2, ..., t_N = T$, where $t_n = t_0 + nh$, $n = 0, 1, 2, ..., N$. Note that $h = t_{n+1} - t_n$. We develop a *recursive formula* that gives the approximation X_{n+1} at time t_{n+1} in terms of the previous approximation X_n at time t_n. Using this formula we can march successively to compute the desire approximate values X_0, X_1, X_2, $X_3, \ldots, X_N$.

Let $x = x(t)$ denote the exact solution to the initial value problem. Then $x'(t) = f(t, x(t))$ for all t in $[0, T]$. We integrate this equation from t_n to t_{n+1} to get

$$\int_{t_n}^{t_{n+1}} x'(t)dt = \int_{t_n}^{t_{n+1}} f(t, x(t))dt. \tag{6.4}$$

The left side can be calculated by the fundamental theorem of calculus, and the right side can be approximated using the *left-hand rule* from calculus for estimating integrals. We get

$$x(t_{n+1}) - x(t_n) = hf(t_n, x(t_n)) + \mathrm{O}(h^2),$$

where the *error* $\mathrm{O}(h^2)$ denotes a term of *order* h^2, meaning it is proportional to h^2, the step size-squared. Rearranging,

$$x(t_{n+1}) = x(t_n) + hf(t_n, x(t_n)) + \mathrm{O}(h^2).$$

If we denote the approximation of $x(t_n)$ by X_n and neglect the apparently small error term, then we can write

$$X_{n+1} = X_n + hf(t_n, X_n), \quad n = 0,\ 1,\ 2,\ 3, \ldots N, \tag{6.5}$$

which is the **Euler method**[1]. To start the method at $n = 0$, we take $X_0 = x_0$, the initial condition. Then we use (6.5) to compute X_1, X_2, and so on.

To summarize, equation (6.5) provides an algorithm for calculating approximations X_1, X_2, X_3, and so on, recursively, at times t_1, t_2, t_3,... . The discrete approximation consisting of the values X_0, X_1, X_2, $X_3, \ldots$, is called a **numerical solution** to the initial value problem. These discrete values approximate the graph of the exact solution, and often they are connected by line segments to obtain a continuous curve.

The error term, $\mathrm{O}(h^2)$, is called the **local truncation error**; it is the error introduced in replacing the integral by an approximation on one subinterval. We say the local truncation error is *order* h^2, or second order. If we make an order h^2 error over one step, from t_n to t_{n+1}, then the **cumulative error**, or global discretization error, taken over all N steps, from t_0 to t_N, should be on the order N times $\mathrm{O}(h^2)$, or $\mathrm{O}(h)$, since $N = (T - t_0)/h$. Therefore, applying the Euler method over a bounded interval should give a global error proportional to the step size h at the right endpoint. We say Euler's method is order h, written $\mathrm{O}(h)$.

Example 6.3

Consider the initial value problem

$$x' = t - x, \quad x(0) = 1.$$

We solve the equation numerically over the interval $0 \le t \le 2$, taking $N = 8$ steps; then the step size is $h = (2 - 0)/8 = 0.25$. Here $f(t, x) = x - t$ and $t_n = nh$, $n = 0, 1, 2, \ldots, 8$. The Euler equation (6.5) with step size h is

$$\begin{aligned} X_{n+1} &= X_n + h(t_n - X_n) \\ &= X_n + h(nh - X_n) \\ &= X_n + 0.25(0.25n - X_n). \end{aligned}$$

Beginning with $X_0 = 1$ we have

$$X_1 = X_0 + (0.25)(0.25 \cdot 0 - X_0) = 1 + (0.25)(-1) = 0.75.$$

Then

$$X_2 = X_1 + (0.25)(0.25 \cdot 1 - X_1) = 0.75 + (0.25)(0.25 \cdot 2 - 0.75) = 0.625.$$

Continuing in this manner we generate the remaining approximations (to 3 decimal places)

$$X_3 = 0.594,\ X_4 = 0.632, \ldots, X_7 = 1.017,\ X_8 = 1.200.$$

[1] The method is named after the Swiss mathematician L. Euler (1707–1783).

We can connect the approximations by straight line segments to generate a continuous curve.

In Figure 6.1 we show a numerical solution obtained in 3 cases, $h = 0.5,\ 0.25,\ h = 0.125$, corresponding to $N = 4, 8, 16$ steps, respectively. We compare them to the exact solution on the interval $[0, 2]$, obtained by the integrating factor method:

$$x(t) = t - 1 + 2e^{-t}.$$

For each approximation, the table shows the value calculated at the endpoint $t = 2$ of the interval compared to the exact solution at that point. The error is the exact value minus the approximate value. Observe that the error is approximately halved as the step size h is halved, which is consistent with the fact that the cumulative error in the Euler method is order h. □

h	$x(2)$	error
0.5	1.125	0.1457
0.25	1.2002	0.0705
0.125	1.2361	0.0346
exact	1.2707	0

Because it is tedious to do numerical calculations by hand, we can program a calculator or write a simple set of instructions for a computer algebra system to do the work for us. Most calculators and computer algebra systems have built-in programs that implement the Euler algorithm, as well as others, automatically. In Example 6.3, for illustration only, the smallest step size is $h = 0.125$, requiring 16 steps. But a typical step size in a real problem may be $h = 0.001$, or even smaller. Furthermore, in science and engineering we often write simple programs that implement recursive algorithms like the Euler method; that way we know the skeleton of our calculations, which is often preferred to plugging into an unknown black box containing a canned program.

There is another insightful, geometrical way to understand the Euler algorithm using the slope field of the differential equation. Imagine we are at the position (t_n, X_n) and want to calculate the next value X_{n+1}. See (Figure 6.2). Draw the tangent line to the solution curve at (t_n, X_n), which has equation

$$x - X_n = f(t_n, X_n)(t - t_n).$$

This straight line is in the direction of the slope field at the point (t_n, X_n). Go along that line until you hit the vertical line $t = t_{n+1}$; the X value on that vertical line is X_{n+1}, which satisfies

$$X_{n+1} - X_n = f(t_n, X_n)(t_{n+1} - t_n).$$

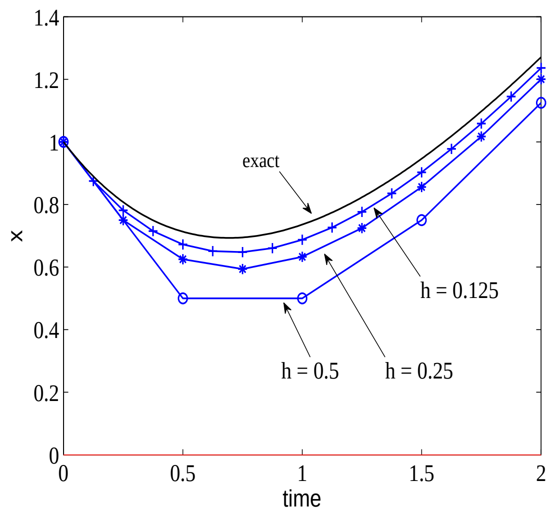

Figure 6.1 Three numerical solutions using the Euler method with $h = 0.5$, $0.25,\ 0.125$. They are compared to the exact solution. We expect the cumulative error at $t = 2$ to be order of the step size h.

Because $h = t_{n+1} - t_n$, the last equation is just Euler's method. In summary, the Euler method computes approximate values by moving in the direction of the slope field at each approximate point. This explains why the the numerical solution in Example 6.3 (Figure 6.1) lags behind the exact solution, which is concave up. The Euler scheme is called an **explicit method** because it permits the calculation of X_{n+1} *directly* from X_n.

The Modified Euler Method

The Euler algorithm is the simplest method for numerically approximating the solution to a differential equation. To obtain a more accurate method, we can approximate the integral on the right side of (6.4) by the *trapezoidal rule*, which is more accurate than the left-hand rule. Then

$$x(t_{n+1}) - x(t_n) = \frac{h}{2}[f(t_n, x(t_n)) + f(t_{n+1}, x(t_{n+1}))] + \mathrm{O}(h^3),$$

where now the local truncation error is order h^3. Then, discarding the error and letting X_n denote the approximation of $x(t_n)$, we obtain

$$X_{n+1} = X_n + \frac{h}{2}[f(t_n, X_n) + f(t_{n+1}, X_{n+1})]. \tag{6.6}$$

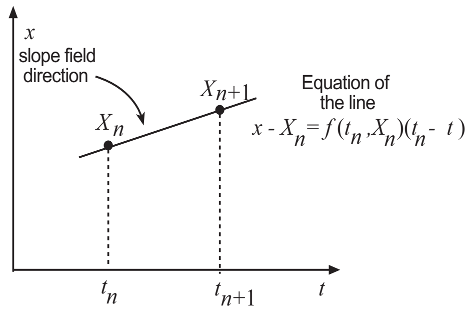

Figure 6.2 The Euler method. Geometrically, if (t_n, X_n) is known, where X_n is the nth approximation, then the $(n+1)$st approximation X_{n+1} is calculated by marching along the slope field from the point (t_n, X_n) to the line $t = t_{n+1}$.

This difference equation is not as simple as it appears. It does not give the X_{n+1} explicitly in terms of the X_n because the X_{n+1} is tied up in a possibly nonlinear term on the right side. Such an equation is called an **implicit equation**. At each time step we have to solve a nonlinear algebraic equation for the X_{n+1}. We can do this numerically, which is very time consuming. Does it pay off in more accuracy? The answer is yes. The Euler algorithm makes a cumulative error over an interval proportional to the step size h, whereas the implicit method makes a cumulative error of order h^2. Observe that $h^2 < h$ when h is small.

A better approach which avoids having to solve a nonlinear algebraic equation at each step is to replace the X_{n+1} on the right side of (6.6) by the X_{n+1} calculated by the Euler method. That is, we compute a *predictor*

$$\widetilde{X}_{n+1} = X_n + hf(t_n, X_n), \tag{6.7}$$

and then use that to calculate a *corrector*

$$X_{n+1} = X_n + \frac{h}{2}[f(t_n, X_n) + f(t_{n+1}, \widetilde{X}_{n+1})]. \tag{6.8}$$

This algorithm is an example of a **predictor–corrector method**, and again the cumulative error is proportional to h^2, an improvement to the Euler method. This is not insignificant; if $h = 0.1$ for example, then $h^2 = 0.01$. This method is called the **modified Euler method**. Some authors refer to it as the second-order Runge–Kutta method.

The modified Euler method gives the new value X_{n+1} as the average of two slopes: the slope at (t_n, X_n) and the slope at $(t_{n+1}, \widetilde{X}_{n+1})$.

Example 6.4

Consider the IVP

$$x' = -2tx + \sqrt{t}, \quad x(0) = 4.$$

This initial value problem is linear and can be solved by integrating factors to obtain

$$x(t) = e^{-t^2}\left(4 + \int_0^t \sqrt{s}\, e^{s^2}\, ds\right).$$

We set up the modified Euler algorithm. The recursion is given by the predictor,

$$\widetilde{X}_{n+1} = X_n + h\left(-2t_n X_n + \sqrt{t_n}\right),$$

and the corrector

$$X_{n+1} = X_n + \frac{h}{2}\left[(-2t_n X_n + \sqrt{t_n}) + (-2t_{n+1}\widetilde{X}_{n+1} + \sqrt{t_{n+1}})\right].$$

Here, $t_n = nh$ and $t_{n+1} = (n+1)h$. Starting with $t_0 = 0$ and $X_0 = 4$, we compute X_1, X_2, $X_3, \ldots$ recursively, in a loop, by these formulas. □

6.2.2 The Runge–Kutta Method

The Euler and modified Euler methods are two of many numerical constructs to solve differential equations. Because solving differential equations is so important in science and engineering, and because real-world models are usually quite complicated, great efforts have gone into developing accurate efficient methods. The most popular algorithm and workhorse of the subject is the highly accurate, explicit, fourth-order **Runge–Kutta method**, where the cumulative error over a bounded interval is proportional to h^4. The Runge–Kutta update formula is

$$X_{n+1} = X_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$\begin{aligned}
k_1 &= f(t_n, X_n),\\
k_2 &= f(t_n + \frac{h}{2}, X_n + \frac{h}{2}k_1),\\
k_3 &= f(t_n + \frac{h}{2}, X_n + \frac{h}{2}k_2),\\
k_4 &= f(t_n + h, X_n + hk_3).
\end{aligned}$$

We do not derive the formulas here, but we note that the new value is a weighted average of the four slopes $k_1, \ldots, k_4$ taken at various points in the interval. The only issue is that four function evaluations are required; but with current-day processor speeds that has become a moot point. The Runge–Kutta method is built-in on computer algebra systems and scientific calculators, and it is quite easy to program.

Observe that the order of the error makes a huge difference in the accuracy. If the step $h = 0.1$, then the cumulative errors over an interval for the Euler, modified Euler, and Runge–Kutta methods are proportional to 0.1, 0.01, and 0.0001, respectively.

Error Analysis

Readers who want a detailed account of the errors involved in numerical algorithms should consult a text on numerical analysis or on numerical solution of differential equations. Overall, there are three sources of errors in a finite difference scheme: truncation, the instability of the difference scheme itself, and roundoff error. We saw the source of the discretization, or truncation error, which arises from approximating a continuous problem by a discrete one. We have not discussed at all the issue of stability. Suffice it to say that the discrete algorithm *itself* may be unstable. That is, an error at the first step can be amplified significantly as it is propagated forward in time by the difference scheme. Finally, roundoff error is the error that a computer makes when it represents real numbers, for example, π, by rational numbers. For example, if $x(t_n)$ is the exact solution at t_n, we represent it with an approximation X_n found from a difference formula. The error $x(t_n) - X_n$ is due to truncation. However, when we implement the algorithm on a computer, we compute actual numerical values $\overline{X}_n$. The error $X_n - \overline{X}_n$ is the roundoff error. Thus, the total error is

$$x(t_n) - \overline{X}_n = (x(t_n) - X_n) + (X_n - \overline{X}_n),$$

which is the sum of the truncation and the roundoff error. In this formula the numerical stability of the scheme is not considered.

Finally, we comment that some differential equations are *stiff* in nature. That is, there are small intervals of time where the solution changes dramatically from one value to another. An example is a chemical reaction that goes slowly at the beginning but then goes to completion extremely rapidly. It should be clear that approximating the solution by simple slope fields cannot keep up with the rapid changes. Even the very accurate Runge–Kutta fails. Fortunately, *stiff methods* have been developed that accurately approximate these types of solutions. These methods are included in the codes in various software packages.

6.3 Systems of Equations

The numerical methods introduced for a single equation easily extend to systems of equations. We now catalog the results. Consider the two-dimensional system

$$x' = f(t, x, y),$$
$$y' = g(t, x, y),$$

with initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0.$$

We want to obtain a numerical solution on the interval $t_0 \le t \le T$.

The first step is to discretize the interval as before by defining the time steps by

$$t_n = t_0 + nh, \;\; n = 0, 1, 2, \ldots, N,$$

where $h = (T - t_0)/N$ is the step size and N is the number of steps. Again, we use the notation X_n and Y_n to denote the approximate values of $x(t_n)$ and $y(t_n)$, respectively. We can summarize the methods as follows.

Euler Method

$$X_{n+1} = X_n + hf(t_n, X_n, Y_n),$$
$$Y_{n+1} = Y_n + hg(t_n, X_n, Y_n).$$

Modified Euler method

We first compute the *predictors* using the Euler method,

$$\widetilde{X}_{n+1} = X_n + hf(t_n, X_n, Y_n), \quad \widetilde{Y}_{n+1} = Y_n + hg(t_n, X_n, Y_n),$$

and then use those to calculate the *correctors*

$$X_{n+1} = X_n + \frac{h}{2}[f(t_n, X_n, Y_n) + f(t_{n+1}, \widetilde{X}_{n+1}, \widetilde{Y}_{n+1})],$$
$$Y_{n+1} = Y_n + \frac{h}{2}[g(t_n, X_n, Y_n) + g(t_{n+1}, \widetilde{X}_{n+1}, \widetilde{Y}_{n+1})].$$

Runge–Kutta method

First we compute the values of the eight slopes:

$$\begin{aligned}
k_{11} &= f\left(t_n, X_n, Y_n\right) \\
k_{21} &= g\left(t_n, X_n, Y_n\right) \\
k_{12} &= f\left(t_n + \tfrac{h}{2}, X_n + \tfrac{h}{2}k_{11},\ Y_n + \tfrac{h}{2}k_{21}\right) \\
k_{22} &= g\left(t_n + \tfrac{h}{2}, X_n + \tfrac{h}{2}k_{11}, Y_n + \tfrac{h}{2}k_{21}\right) \\
k_{13} &= f\left(t_n + \tfrac{h}{2}, X_n + \tfrac{h}{2}k_{12}, Y_n + \tfrac{h}{2}k_{22}\right) \\
k_{23} &= g\left(t_n + \tfrac{h}{2}, X_n + \tfrac{h}{2}k_{12}, Y_n + \tfrac{h}{2}k_{22}\right) \\
k_{14} &= f\left(t_n, X_n + hk_{13}, Y_n + hk_{23}\right) \\
k_{24} &= g\left(t_n, X_n + hk_{13}, Y_n + hk_{23}\right)
\end{aligned}$$

Then we compute the next approximation using weighted averages of these slopes,

$$\begin{aligned}
X_{n+1} &= X_n + \frac{h}{6}\left(k_{11} + 2k_{12} + 2k_{13} + k_{14}\right), \\
Y_{n+1} &= Y_n + \frac{h}{6}\left(k_{21} + 2k_{22} + 2k_{23} + k_{24}\right).
\end{aligned}$$

The global errors in these three methods do not change, and remain order h, h^2, and h^4, respectively. It should be clear that the same procedures can be extended to several equations in several unknowns.

There are many ways to implement these algorithms in a variety of computer algebra systems. Many of the solutions and plots in the text were created with MATLAB. Appendix B contains a summary of important commands. Scientific calculators also implement the Euler and Runge–Kutta methods, but without the high quality graphics.

EXERCISES

1. **(a)** Use the Euler method and the modified Euler method to numerically solve the initial value problem

$$x' = 0.25x - t^2, \quad x(0) = 2,$$

on the interval $0 \le t \le 2$ using a step size $h = 0.25$. Compare them graphically, and compare the final values $x(2)$ at the final $t = 2$ value. **(b)** Perform calculations with $h = 0.1$, $h = 0.01$, and $h = 0.001$, and confirm that the cumulative error at $t = 2$ is roughly order h for the Euler method and order h^2 for the modified Euler method.

2. Use the Euler method to solve the initial value problem $x' = x\cos t$, $x(0) = 1$ on the interval $0 \le t \le 20$ with 50, 100, 200, and 400 steps. Compare with the exact solution and comment on the accuracy of the numerical algorithm.

3. A population of bacteria, given in millions of organisms, is governed by the law
$$x' = 0.6x\left(1 - \frac{x}{K(t)}\right), \quad x(0) = 0.2,$$
where in a periodically varying environment the carrying capacity is $K(t) = 10 + 0.9\sin t$, and time is given in days. Plot the bacteria population for 40 days. Use the Euler method.

4. Consider the initial value problem on $0 \le t \le 5$ given by
$$\frac{dx}{dt} - x = \frac{11}{8}e^{-t/3}, \quad x(0) = -\frac{33}{32}.$$
(a) Use the Euler method with $h = 0.001$ to approximate the solution. **(b)** Determine the validity of the approximation by finding the exact solution. **(c)** Make the same calculation using the Runge–Kutta method. Is this approximation accurate? **(d)** Explain the findings or your numerical experiments. Hint: Examine the slope field.

5. Suppose the temperature inside your winter home is 68 degrees at 2:00 P.M. and your furnace then fails. If the outside temperature has an hourly variation over each day given by $15 + 10\cos(\pi t/12)$ degrees (where $t = 0$ represents 2:00 P.M.), and you notice that by 10:00 P.M. the inside temperature is 57 degrees, what will be the temperature in your home the next morning at 6:00 A.M.? Sketch a plot showing the temperature inside your home and the outside air temperature.

6. Consider the initial value problem $x' = 5x - 6e^{-t}$, $x(0) = 1$. Find the exact solution and plot it on the interval $0 \le t \le 3$. Next use the Euler method with $h = 0.1$ to obtain a numerical solution. Explain the results of this numerical experiment.

7. Consider the IVP
$$x' = x^2, \quad x(0) = 0.99,$$
which has the solution
$$x(t) = \frac{99}{100 - 99t}.$$
Thus, $x(1) = 99$. **(a)** Use the Euler method to approximate $x(1)$ for step sizes $h = 0.1, 0.05, 0.01, 0.005, 0.001$, and 0.0005. Comment on its accuracy. **(b)** Repeat the calculation using the modified Euler and the Runge–Kutta methods. What do you conclude?

8. For the simple differential equation $x' = f(t),\ x(0) = x_0$, show that the Runge–Kutta method reduces to Simpson's rule for integration.

9. Numerically solve the initial value problem

$$q' = -q - 5e^{-t}\sin 5t, \quad q(0) = 1$$

on the interval $0 \le t \le 3$. Note: This is the RC circuit equation ($R = 1$, $C = 1$) with an oscillating, decaying emf; $q = q(t)$ is the charge on the capacitor.

10. (*Resistive heating*) Consider an RCL circuit where the resistance depends on temperature (a thermistor). Assume the resistor is a rectangular solid made of copper having length l and cross-sectional area A. Its resistance is $R(T) = \rho(T)l/A$, where $\rho(T)$ is the resistivity, which depends on the temperature T via $\rho(T) = \rho_0(1 + \alpha T)$, $\alpha > 0$. Here, ρ_0 is the resistivity at $T = 0$ degrees C.

   a) State why

$$L\frac{d^2I}{dt^2} + R_0(1 + \alpha T)\frac{dI}{dt} + \frac{I}{C} = 0, \qquad R_0 \equiv \frac{\rho_0 l}{A}.$$

   b) Show that *Joule's law*, that is, the rate of change of thermal energy equals the rate of ohmic heating, is expressed by

$$c\delta Al\frac{dT}{dt} = R_0(1 + \alpha T)I^2,$$

   where c is the specific heat of the resistor and δ is its density. (For reference and extensions, see J. D. Logan, *Int. J. Math. Sci. Tech.* 30(6), 1999.)

   c) Numerically calculate and plot the temperature of the resistor and the current in the circuit up to the melting point of the copper resistor, 1083 deg. Let $I(0) = 0$, $Q(0) = Q_0$, $T(0) = T_0$ denote the initial current, charge, and temperature of the resistor. Take $L = 40(10)^{-8}$ henrys, $C = 56(10)^{-6}$ farads, $Q_0 = 10^4$ coulombs, $T_0 = 20$ deg, $A = 10^{-9}$ m^2, $l = 0.02$ m, $\alpha = 0.004$, $c = 392.9$ J/(kg$\cdot$ deg), $\delta = 8890$ kg/m^3, $\rho_0 = 1.63(10)^{-8}$ ohm-m.

A
Review and Exercises

This supplement begins with a brief review of first- and second-order equations, including exercises and their solutions. This material corresponds to Section 3.7 of the 2nd edition of the text and the first two chapters of the current edition.

The second part of this supplement features additional exercises corresponding to each chapter in the text. Many of the questions come from actual examinations, and they provide a problem bank for instructors and an opportunity for students to review and assess their skills.

A.1 Review Material

One way to think about learning and solving differential equations is in terms of pattern recognition. Although this is a compartmental way of thinking, it does help our learning process. When faced with a differential equation, what do we do? The first step is to recognize what type it is. It is like a pianist recognizing a certain sequence of notes in a complicated musical piece and then playing those notes easily because of repeated practice. In differential equations we practice identifying equations and learning solution techniques that solve those equations.

- **Antiderivatives.** The simplest differential equation is

$$x' = g(t).$$

The solution $x = x(t)$ is the antiderivative of $g(t)$, which we write as $x(t) = \int g(t)dt + C$. Often we write the solution as an integral with a variable upper limit of integration,

$$x(t) = \int_a^t g(s)ds + C.$$

- **Separable equations.** A separable equation has the form

$$\frac{dx}{dt} = g(t)f(x),$$

where the right side is a product of functions of the dependent and independent variables. Immediately we separate variables and integrate to get

$$\int \frac{1}{f(x)}dx = \int g(t)dt + C.$$

- **Linear first-order equations.** To solve a first order linear equation

$$x' + p(t)x = q(t),$$

multiply the equation by the integrating factor $e^{\int p(t)dt}$ which reduces it to

$$\left(xe^{\int p(t)dt}\right)' = q(t)e^{\int p(t)dt}.$$

This reduced equation can be integrated immediately to get

$$xe^{\int p(t)dt} = \int q(t)e^{\int p(t)dt}dt + C,$$

giving

$$x(t) = e^{-\int p(t)dt}\left(\int q(t)e^{\int p(t)dt}dt + C\right).$$

- **Bernoulli equations.** The transformation $y(t) = x(t)^{1-n}$ reduces the Bernoulli equation

$$x' + p(t)x = q(t)x^n$$

to a linear equation for $y = y(t)$.

- **Homogeneous equations.** An equation of the form

$$x' = f\left(\frac{x}{t}\right)$$

can be reduced to a separable equation using the change of variables $y(t) = x(t)/t$.

- **Exact equations.** The differential equation

$$f(t, x) + g(t, x)x' = 0$$

is exact if $f_x = g_t$. The solution is given by $H(t, x) = C$ where H is found by solving $H_t = f(t, x)$ and $H_x = g(t, x)$.

- **Autonomous equations.** These equations have the form

$$x' = f(x),$$

where the right side depends only on x. They may be solved by separation of variables, but we commonly treat them using qualitative methods to understand the behavior of solutions. First we plot $f(x)$ versus x. The equilibria are constant solutions satisfying $f(x) = 0$; the equilibria are points where $f(x)$ crosses the x axis. Next we construct a phase line diagram by placing arrows on the x axis to denote the direction that x follows as t increases; where $f(x) > 0$ the arrows go to the right, and where $f(x) < 0$ the arrows go to the left. This determines the stability of each equilibrium. The analytic condition to guarantee stability at an equilibrium x^* is $f'(x^*) < 0$. If $f'(x^*) > 0$ the equilibrium is unstable, and if $f'(x^*) = 0$ we have no information.

- **Second-order equations.** There are two second-order linear equations that can be solved simply: equations with constant coefficients and Cauchy–Euler equations,

$$ax'' + bx' + cx = 0, \qquad at^2x'' + btx' + cx = 0.$$

The former has solutions of the form $x(t) = e^{\lambda t}$, where λ satisfies the characteristic equation $a\lambda^2 + b\lambda + c = 0$, and the Cauchy–Euler equation has solutions of the form $x = t^m$, where m satisfies the characteristic (indicial) equation $am(m-1) + bm + c = 0$. For both equations we must distinguish cases when the roots of the characteristic equation are real and unequal, real and equal, or complex. In the complex root case we use Euler's formula to obtain two real solutions. In each case we get two independent solutions and the general solution is a linear combination of those two solutions.

- **Nonhomogeneous equations.** Nonhomogeneous equations have the form

$$x'' + p(t)x' + q(t)x = f(t).$$

In all cases the *variation of parameters formula* applies to find the particular solution:

$$x_p(t) = -x_1(t)\int \frac{x_2(t)f(t)}{W(t)}dt + x_2(t)\int \frac{x_1(t)f(t)}{W(t)}dt,$$

where $x_1(t)$ and $x_2(t)$ are basic solutions to the homogeneous problem, and $W(t) = x_1x_2' - x_1'x_2$ is the Wronskian. In the variable coefficient case it may not be possible to find two independent solutions of the homogeneous equation. (If we know one solution, the *reduction of order* method may lead to a second one.) For the constant coefficient case, a particular solution can be more easily found using *undetermined coefficients* (judicious guessing) when $f(t)$ is a polynomial, exponential, sine or cosine, or sums and products of those basic forms. The basic structure theorem holds for all linear nonhomogeneous equations: *the general solution is the sum of the general solution to the homogeneous equation and a particular solution of the nonhomogeneous equation.*

- **Conservation Laws.** Equations involving Newton's second law,

$$mx'' = F(x, x')$$

are usually handled by transforming the equation into to a system $x' = y$, $y' = m^{-1}F(x, y)$. If $F = F(x)$, depending only on position, the equation is conservative and there is a potential function $V(x) = -\int F(x)dx$. Then, dividing the two first order equations, $x' = y,\ y' = m^{-1}F(x, y)$, and then integrating gives the conservation of energy law,

$$\frac{1}{2}my^2 + V(x) = E,$$

where the constant E is the total energy in the system; E depends on the initial conditions and is found by substituting $x = x(0)$ and $y = y(0)$ into the conservation law. We can solve for the velocity y and plot velocity vs. position. Alternately, we can replace y by dx/dt and separate variables to obtain $x = x(t)$ in implicit integral form:

$$t = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{d\xi}{\sqrt{E - V(\xi)}} + t_0.$$

Review Exercises

1. Identify each of the differential equations and find the general solution. Some of the solutions may contain an integral.

a) $2x'' + 5x' - 3x = 0$.

b) $x' - Rx = 0$, where $R > 0$.

c) $x' = \cos t - x\cos t$.

d) $x' - 6x = e^t$.

e) $x'' = -\frac{2}{t^2}x$.

f) $x'' + 6x' + 9x = 5\sin t$.

g) $x' = -8t + 6$.

h) $x'' + x = t^2 - 2t + 2$

i) $x' + x - tx^3 = 0$.

j) $2x'' + x' + 3x = 0$.

k) $x'' = (x')^3$.

l) $tx' + x = t^2x^2$.

m) $x'' = -3x^2$.

n) $tx' = x - \frac{t}{2}\cos^2\left(\frac{2x}{t}\right)$.

o) $x''' + 5x'' - 6x' = 9e^{3t}$.

p) $(6tx - x^3) = -(4x + 3t^2 - 3tx^2)x'$.

2. Solve the initial value problem $x' = x^2 \cos t$, $x(0) = 2$, and find the interval of existence.

3. Solve the initial value problem $x' = -\frac{4}{t}x + t$, $x(1) = 2$, and find the interval of existence.

4. For all cases, find the equilibrium solutions for $x' = (x - a)(x^2 - a)$, where a is a real parameter, and determine their stability. Summarize the information on a bifurcation diagram.

5. A spherical water droplet loses volume by evaporation at a rate proportional to its surface area. Find its radius $r = r(t)$ in terms of the proportionality constant and its initial radius r_0.

6. A population is governed by the law

$$p' = rp\left(\frac{K - p}{K + ap}\right)$$

where r, K, and a are positive constants. Find the equilibria and their stability. Describe, in words, the dynamics of the population.

7. Use the variation of parameters method to find a particular solution to $x'' - x' - 2x = \cosh t$.

8. Solve $x' = 4tx - \frac{2x}{t}\ln x$ by making the substitution $y = \ln x$.

9. The equation $(1 - t^2)x'' - 2tx' + 2x = 0$ has a solution $x(t) = t$. Find a second linearly independent solution of the form $tw(t)$, for some function w. What is the general solution of the equation?

■ Solutions to Review Exercises

1. Describe the type of equation and solve.

a) (Second-order linear) The characteristic equation is $2\lambda^2 + 5\lambda - 3 = 0$ with eigenvalues $1/2$ and -2. The general solution is $x(t) = c_1e^{t/2} + c_2e^{-2t}$.

b) (Decay equation) $x(t) = Ce^{Rt}$.

c) (Separable) $x(t) = 1 - Ce^{\sin t}$.

d) (First-order linear) The integrating factor is e^{-6t}. The solution is $x(t) = -e^t/5 + Ce^{6t}$.

e) (Cauchy–Euler) The indicial equation is $m(m-1)+2 = 0$, which has complex roots. The solution is $x(t) = t^{-1/2}(c_1 \cos[(\sqrt{7}/2)\ln t] + c_2 \sin[(\sqrt{7}/2)\ln t])$.

f) (Second-order nonhomogeneous) $x_h(t) = e^{-3t}(c_1 + c_2 t)$. The particular solution takes the form $x_p(t) = A\cos t + B\sin t$.

g) (Antiderivative) $x(t) = -4t^2 + 6t + C$.

h) (Second-order nonhomogeneous) The homogeneous solution is $x_h(t) = c_1 \cos t + c_2 \sin t$. A particular solution has the form $x_p(t) = At^2 + Bt + C$.

i) (Bernoulli) Make the substitution $y = 1/x^2$, $y' = -(2/x^3)x'$ to transform it to a linear equation in $y(t)$.

j) (Second-order) The characteristic equation has roots $-1/4 \pm i\sqrt{23}/4$. The solution is $x(t) = e^{-t/4}(c_1 \cos \sqrt{23}\, t/4 + c_2 \cos \sqrt{23}/, t/4)$.

k) (Newton's law) Make the substitution $y = x'$ to get the separable equation $y' = y^3$. This separates to give $y = \pm 1/\sqrt{C_1 + 2t}$. Therefore, $x(t) = \int \pm 1/\sqrt{C_1 + 2t}\; dt + C_2$, which can be integrated to find $x(t)$. (Hint: Make the substitution $w = C_1 + 2t$, $dw = 2dt$.)

l) (Bernoulli) Let $y = 1/x$, $y' = -1/x^2 x'$ to transform the equation into a linear equation for $y(t)$.

m) (Conservative equation) The potential energy is $V(x) = x^3$ and we have $y^2/2 + x^3 = E$. Letting $y = x'$, separating variables, and integrating gives

$$t + C = \pm \int \frac{1}{\sqrt{E - 2x^3}}\, dx.$$

n) (Homogeneous) Make the substitution $y = x/t$ to get the separable equation $ty' = -\frac{1}{2}\cos^2 2y$. Integrate to get $-\tan 2y = \ln t + C$, implicitly. Now solve for y.

o) (Third-order nonhomogeneous) The homogeneous solution is $x_h(t) = c_1 + c_2 e^t + c_3 e^{-6t}$ and the particular solution takes the form $x_p(t) = Ae^{3t}$.

p) (Exact) Write the equation as $(6tx - x^3)dt + (4x + 3t^2 - 3tx^2)dx = 0$. The equation is exact. Therefore the solutions are $H(t,x) = C$ where

$H_t = 6tx - x^3$, $H_x = 4x + 3t^2 - 3tx^2$. Integrate the first equation with respect to t, obtaining an arbitrary function $\phi(x)$; then take the x derivative of the expression to compare to the second equation. This determines the arbitrary function $\phi(x) = 2x^2$. Finally we get $H(t,x) = 3t^2x - tx^3 + 2x^2 = C$.

2. The equation is separable and has the solution $x(t) = 1/(0.5 - \sin t)$. This is valid as long as $\sin t \neq 0.5$. This give the range $-7\pi/x < t < \pi/6$.
3. The equation is first order linear with integrating factor t^4. The solution is $x(t) = t^2/6 + 11/6t^4$. This is valid on $(0, +\infty)$.
4. The model is $V' = -k4\pi r^2$. Since $V = 4\pi r^3/3$, we get $V' = 4\pi r^2 r'$. Thus $r' = -k$, giving $r(t) = r_0 - kt$.
5. The equilibria are $p = 0$ (unstable) and $p = K$ (stable). Note $p' > 0$ for $0 < p < K$ and $p' < 0$ for $p > K$.
6. The solutions to the homogeneous equation are e^{2t}, e^{-t}. The Wronskian is $W(t) = -3e^t$. By the variation of parameters formula we have

$$x_p(t) = -e^{2t}\int \frac{e^{-t}\cosh t}{W(t)}dt + e^{-t}\int \frac{e^{2t}\cosh t}{W(t)}dt.$$

Replace $\cosh t$ by $(e^t + e^{-t})/2$, simplify, and compute the integrals.

7. Let $y = \ln x$. Then $y' = x'/x$ and the differential equation becomes $y' + (2/t)y = 4t$, which is linear with integrating factor t^2. Its solution is $y(t) = t^2 + C/t^2$. So $x(t) = e^{y(t)}$.
8. Substitute $x = tw(t)$ into the differential equation to get $(t - t^3)w'' + (2 - 4t^2)w' = 0$. Solve the equation to get a second solution

$$x(t) = -1 + \frac{t}{2}\ln\frac{1+t}{1-t}.$$

The general solution is a linear combination of the the two independent solutions.

A.2 Supplementary Exercises

■ Chapter 1 Exercises

1. Find the function $x = x(t)$ that solves the initial value problem $x' = (\ln t + t^2)/t, \quad x(1) = 0$.

2. A particle of mass 2 moves in one dimension with *acceleration* given by $3 - v(t)$, where $v = v(t)$ is its velocity. If its initial velocity is $v = 1$, when, if ever, is the velocity equal to two?

3. Find $y'(t)$ if
$$y(t) = t^2 \int_1^t \frac{1}{r} e^{-r} dr.$$

4. Consider the autonomous equation
$$\frac{dx}{dt} = -(x-2)(x-4)^2.$$
Find the equilibrium solutions, sketch the phase line, and indicate the type of stability of each equilibrium solutions.

5. Find the solution of the initial value problem
$$y' - \frac{2}{t+1} y = (t+1),\ y(0) = 3.$$

6. Find the (implicit) solution of the DE
$$x' = \frac{1+t}{3tx^2 + t}$$
that passes through the point $(1,1)$.

7. A mass of $m = 1$ gm is subjected to a positive force proportional to the square root of the velocity; the initial velocity is 3 cm/sec. Find the velocity as a function of time and sketch a time series plot for $t \geq 0$.

8. An RC circuit has $R = 1$, $C = 2$. Initially the voltage drop across the capacitor is 2 volts. For $t > 0$ the applied voltage (emf) in the circuit is $b(t)$ volts. Write down an IVP for the *voltage* across the capacitor and find a formula for it.

9. Consider the autonomous equation
$$\frac{dP}{dt} = (P-h)(P^2 - 2P), \quad h > 0.$$
Clearly, $P = h$ is an equilibrium. Use the derivative criterion to determine the values of h for which this equilibrium is unstable.

10. A thermometer has been stored in a room whose temperature is 75 degrees. Five minutes after being taken outdoors it reads 65 degrees. After another five minutes it reads 60 degrees. What is the outdoor temperature?

11. Consider the differential equation

$$\frac{dx}{dt} = (t^2+1)x - t^2.$$

a) In the tx plane sketch the set of points where the slope field is zero.

b) Consider the initial value problem consisting of the differential equation and initial condition $x(1) = 3$. State precisely why you are guaranteed that the IVP has a unique solution in some small open interval containing $t = 1$.

12. Consider the differential equation

$$x' = x\left(1 - \frac{x}{t}\right).$$

Plot the set of points in the tx plane where the slope field has value 1. You may use a calculator.

13. Consider the model $x' = \lambda^2 x - x^3$, where λ is a parameter. Draw the bifurcation diagram (equilibria solutions versus the parameter) and determine analytically the stability (stable or unstable) of the branch in the first quadrant.

14. Find two different solutions of the differential equation

$$t^2x'' - 12x = 0$$

having the form $x(t) = t^m$. (That is, determine value(s) of m for which t^m is a solution.)

15. Find an explicit analytic formula for the solution to the initial value problem

$$\frac{dx}{dt} = 2te^{-t^2}, \quad x(0) = 1.$$

16. Solve the initial value problem

$$\frac{dx}{dt} - \frac{2}{t}x = 3, \qquad x(1) = 4.$$

17. A roasting chicken at room temperature (70 deg) is put in a 325 deg oven to cook. The heat loss coefficient for chicken meat is 0.4 per hour. Set up an initial value problem for the temperature $T(t)$ of the chicken at time t. Set up only but do not solve.

18. Consider a population model governed by the autonomous equation

$$p' = \sqrt{2}\,p - \frac{4p^2}{1+p^2}.$$

a) Sketch a graph of the growth rate p' versus the population p, and sketch the phase line.

b) Find the equilibrium populations and determine their stability.

19. Consider the initial value problem

$$(1-t^2)u^2 + \cos t \frac{du}{dt} = 0, \; u(1) = 10.$$

Find the largest open interval of t-values on which this IVP is guaranteed to have a continuous solution.

20. An RC circuit with no emf has an initial charge of q_0 on the capacitor. The resistance is $R = 1$ and the capacitance is $C = 1/2$. Set up an initial value problem for the charge Q on the capacitor and solve it to find $Q = Q(t)$.

21. An autonomous differential equation is given by

$$\frac{dy}{dt} = (y^2 - 4)(a - y)^3,$$

where a is a fixed constant with $a > 6$.

a) Find all equilibrium solutions and draw the phase line diagram. Label all axes with "arrows" appropriately placed on the phase line.

b) Draw an approximate graph of the solution curve $y = y(t)$ when the initial condition is $y(0) = 5$.

■ Chapter 2 Exercises

1. Find the general solution to the equation $x'' + 3x' - 10x = 0$.

2. A mass of 2 kg is hung on a spring with stiffness (spring constant) $k = 3$ N/m. After the system comes to equilibrium, the mass is pulled downward 0.25 m and then given an initial velocity of 1 m/sec. What is the amplitude of the resulting oscillation?

3. Find the general solution to the linear differential equation

$$x'' - \frac{1}{t}x' + \frac{2}{t^2}x = 0.$$

4. Find the general solution $x = x(t)$ of the damped spring–mass equation

$$2x'' + x' + \frac{3}{32}x = 0.$$

5. Find the *general solution* $x = x(t)$ to the differential equation

$$2t^2x'' + 5tx' - 2x = 0.$$

6. Find the *general solution* $y = y(t)$ to the 4th order differential equation

$$y^{(4)} - y = 3t^3.$$

7. Find the *general solution* $x = x(t)$ to the differential equation

$$tx'' + x' = 4t.$$

8. For the equation $x'' + 5x = 2t\cos 5t$, what is the natural frequency? What is the *form* that the particular solution $x_p(t)$ takes? (Do not find the constants.)

9. The solution of a second-order, linear, homogeneous equation is $x(t) = 5 + 2e^{-10t}$. What is the equation?

10. Find the general solution of the fourth-order differential equation

$$x'''' + 16x'' = 0.$$

11. Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + \frac{4}{t}\frac{dy}{dt} + \frac{2}{t^2}y = 0.$$

12. Consider a damped spring–mass system where $x = x(t)$ is the displacement of the mass from equilibrium. Let m, c, and k denote the mass, damping constant, and spring constant, respectively.

    a) If there is no damping and there is a external forcing function of magnitude $3\cos 5t$, what is the relationship between the mass m and spring constant k for which pure resonance occurs?

    b) If $c = 2$ and $k = 0.1$ and there is no external forcing, what values of the mass m will lead to damped oscillations?

13. Find a particular solution to

$$x'' + x = 7 + 6e^t.$$

14. Transform the following nonlinear Bernoulli equation

$$x' + tx = \frac{1}{t^2x}$$

into a linear equation using a transformation of the dependent variable. DO NOT solve the linear equation.

15. An RCL circuit with no emf is governed by the circuit equation

$$LQ'' + RQ' + \frac{1}{C}Q = 0,$$

where $Q = Q(t)$ is the charge on the capacitor.

a) If the resistance is $R = 8$, shade the region in CL parameter space, or the CL plane (C is the horizontal axis, and L is the vertical) where the solution can be described as "oscillatory decay."

b) What is the decay rate?

c) If $R = 0$, what is the natural frequency of oscillation of the circuit? What is its period?

16. Show that the differential equation x"+kx=f(t) has a particular solution

$$x_p(t) = \frac{1}{k}\int_0^t f(s)\sin k(t-s)ds.$$

17. A function f satisfies the condition $2f'(t) = f(1/t)$ for $t > 0$, with $f(1) = 2$. Letting $x = f(t)$, show that x satisfies the equation $t^2x'' + ptx' + qx = 0$, for some constants p and q. Show that $f(t) = Ct^n$, for some n.

18. The equation $tx'' - x' + (1-t)x = 0$ has a solution of the form $x = e^{mt}$ for some m. Determine the value(s) of m and then find the general solution to the differential equation.

■ Chapter 3 Exercises

1. Find the Laplace transform of $x(t) = e^{-3t}h_2(t)$ using the integral definition of Laplace transform.

2. Find the inverse transform of

$$X(s) = \frac{1}{(s-5)^3}.$$

3. Use the convolution integral to solve the initial value problem

$$x'' + 6x = f(t), \quad x(0) = x'(0) = 0.$$

(Write down the correct integral form.)

4. Solve the initial value problem

$$x' + 2x = \delta_a(t), \quad x(0) = 1,$$

where $\delta_a(t)$ is a unit impulse at some fixed time $t = a > 0$. Sketch a generic plot of the solution for $t \geq 0$.

5. Solve the IVP

$$u' + 3u = \delta_2(t) + h_4(t), \quad u(0) = 1.$$

6. Find the inverse transformation of

$$X(s) = \frac{s}{(s^2 - 10)(s - 5)}$$

using convolution. Write down the appropriate convolution integral, but do not calculate it.

7. Solve the initial value problem using Laplace transforms:

$$x' + 2x = e^{-t} h_3(t), \quad x(0) = 0.$$

■ Chapter 4 Exercises

1. Classify the type and stability of the equilibrium of the system

$$x' = -2x + y,$$
$$y' = -2x.$$

In a phase plane, draw in the nullclines (as dashed lines) and indicate which is which. Then, noting the direction field along the x axis, sketch in a couple of sample orbits.

2. Consider the two-dimensional linear system

$$\mathbf{x}' = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix} \mathbf{x}.$$

a) Find the eigenvalues and corresponding eigenvectors and identify the type of equilibrium at the origin.

b) Write down the general solution.

c) Draw a rough phase plane diagram, being sure to indicate the directions of the orbits.

3. Use eigenvalue methods to find the general solution of the linear system

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} \mathbf{x}.$$

4. Find the equation of the orbits in the xy plane for the system $x' = 4y, \quad y' = 2x - 2$.

5. For the following system, for which values of the constant b is the origin an unstable spiral?

$$x' = x - (b+1)y$$
$$y' = -x + y.$$

6. Classify the equilibrium as to type and stability for the system

$$x' = x + 13y, \quad y' = -2x - y.$$

7. A two-dimensional system $\mathbf{x}x' = A\mathbf{x}$ has eigenpairs

$$-2, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad 1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

   a) If $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, find a formula for $y(t)$ (where $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

   b) Sketch a rough, but accurate, phase diagram.

8. Solve the initial value problem

$$x' = -2x + 2y,$$
$$y' = 2x - 5y,$$
$$x(0) = 3, \quad y(0) = -3.$$

9. Consider

$$x' = 5x - y, \quad y' = -4x - py.$$

   For which values of p is the origin a saddle point?

10. In the xy phase plane, plot the orbit

$$x(t) = 2e^{-t},$$
$$y(t) = -e^{-2t}, \quad -\infty < t < \infty.$$

11. For the the system

$$x' = -2x + 4y,$$
$$y' = -5x + 2y,$$

   sketch a few of the orbits in the phase plane.

12. The general solution of a linear system is

$$x(t) = c_1 e^{-7t} + c_2 e^{-2t},$$
$$y(t) = -c_1 e^{-7t} + \frac{1}{4} c_2 e^{-2t}.$$

(a) State the type and stability of the equilibrium $(0, 0)$, and then draw the linear orbits. (b) Draw on your diagram a few other key orbits, indicating exactly their behavior as they enter the origin. (c) Identify the eigenpairs of the linear system.

■ Chapter 5 Exercises

1. Find the solution $x = x(t)$ of the differential equation

$$x'' = 1 - \frac{x'}{t}.$$

2. A conservative mechanical system is governed by Newton's second law of motion (mass × acceleration = force):

$$2\frac{d^2x}{dt^2} = -xe^{-x^2}.$$

Find the potential energy $V(x)$ of this system for which $V(0) = 0$. Then write down the conservation of energy expression if $x(0) = 0$ and $x'(0) = 1$. Finally, sketch the orbit passing through the point $(0, 1)$.

3. Consider the nonlinear system

$$x' = x(1 - xy),$$
$$y' = 1 - x^2 + xy.$$

   a) Find all the equilibrium solutions.

   b) In the xy plane plot the x and y nullclines.

4. A particle of mass $m = 1$ moves on the x-axis under the influence of a potential $V(x) = x^2(1 - x)$.

   a) Write down Newton's second law, which governs the motion of the particle.

   b) In the phase plane, find the equilibrium solutions. If one of the equilibria is a center, find the type and stability of all the other equilibria.

   c) Draw the phase diagram.

5. Consider the nonlinear system
$$x' = 4x - 2x^2 - xy, \quad y' = y - y^2 - 2xy.$$
Find all the equilibrium points and determine the type and stability of the equilibrium point $(2, 0)$.

6. A particle with mass $m = 2$ moves in one dimension x under the influence of a potential function
$$V(x) = x(x-2).$$
   a) Find the force F on the particle and write down the equation on motion.
   b) If the initial velocity is $y(0) = 0$ and the total energy of the system is $E = 1$, what is the initial position $x(0)$?
   c) Sketch the orbit of the particle in the xy phase plane. (Show its direction).

7. Consider the system
$$x' = xy, \quad y' = 2y.$$
Find a relation between x and y that must hold on the orbits in the phase plane. Sketch the orbits.

8. Consider the system
$$x' = 2y - x, \quad y' = xy + 2x^2.$$
Find the equilibrium solutions. Find and indicate the nullclines and equilibrium solutions on a phase diagram. Draw several interesting orbits.

9. Show that the nonlinear system $x' = y + yx^2, \ \ y' = xy + 2$ has no periodic orbits.

10. Draw the phase diagram for the system $x' = x^2y^3, \ y' = -x^3y^2$.

11. Show that the system $x' = -4x^3(y-2)^2, \ \ y' = 2x^4(2-y)$ is a gradient system, and determine the type and stability of its critical points.

12. Show that the system $x' = 3y^2 - 3x, \ \ y' = 3y - 3x^2$ is a Hamiltonian system. Find the Hamiltonian and sketch the phase diagram.

13. A basic chemostat, or chemical reactor, model has the form
$$x' = -x + \frac{axy}{1+y}, \quad y' = -y + b - \frac{xy}{1+y}, \quad a > 1, \ b > 1/(a-1),$$
where x is the concentration of bacteria in a tank and y is the nutrient in the tank; b is the constant rate that fresh nutrients are supplied to the tank at inflow rate q, and the perfectly stirred mixture is drawn off at the flow rate. Sketch the phase plane diagram indicating the dynamics of the system.

B
MATLAB® Supplement

There is great diversity in differential equations courses with regard to use and choice of technology. For example, MATLAB®, Maple®, *Mathematica*, and others, are all common computer environments used in colleges, universities, and industry, and they perform both symbolic and numerical computations. The same is true for advanced scientific calculators, but not at the scale or with the graphics capability of computer software.

In this appendix we present a menu of useful commands in MATLAB for tasks commonly faced in differential equations. It is not meant to be an introduction or tutorial, but only a summary of the syntax and a few basic commands. The coding algorithms are simple enough that the reader can transcribe them into other systems, like Maple or *Mathematica*. MATLAB topics are listed in many sources on the internet, and they contain complete references with illustrations of all the commands. A good, in-hand reference text is Hunt et. al. (2012). Readers should realize, however, that computer algebra systems are updated regularly and there is danger that some commands will quickly become obsolete as new versions appear. One the positive side, MATLAB code is very logical and similar to a *pseudocode* that may avoid some of the obsolescence.

In MATLAB we can work in the *command window*, or we can write and save text files containing MATLAB commands, either as structured *m-files* or less structured *scripts*. On the next page we begin with a table of useful MATLAB commands.

MATLAB Commands

Command	Instruction
>>	command line prompt
;	semicolon suppresses output
%	for a comment; statements after % are ignored
Ctrl+C	terminate a program
help *topic*	help on MATLAB *topic*
a = 4, b1 = 5	assigns 4 to a and 5 to b1
clear a b1	clears assignments for a and b1
clear all	clears all the variable assignments
x=[0, 3, 6, 9, 12, 15, 18]	row vector (list) assignment
x=0:3:18	defines the same vector as above
x=linspace(0,18,7)	defines the same vector as above
x'	transpose of a row list x
+ - * / ∧	common operations with numbers
sqrt(a)	square root of a
exp(a), log(a)	e^a and $\ln a$
pi	the number π
.* ./ .∧	operations on vectors of same length (with *dot*)
t=0:0.01:5, x=cos(t), plot(t,x)	plots $\cos t$ on $0 \leq t \leq 5$ with spacing 0.01
xlabel('time'), ylabel('state')	labels horizontal and vertical axes
title('Title of Plot')	titles the plot
xlim([a b]), ylim([c d])	sets plot ranges on x and y axes
hold on; hold off	does not plot immediately; releases hold on
for n=1:N,...,end	syntax for a "for-end" loop for n from 1 to N
plot(x)	plots a line graph of a vector x
A=[1 2; 3 4]	defines a matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
x=A\b	solves Ax=b, where b=[α; β] is a column vector
inv(A)	the inverse matrix
A'	transpose of a matrix
det(A)	determinant of A
[V, D] = eig(A)	computes eigenvectors and eigenvalues of A
quad('fun', a, b, tol);	Approximates $\int_a^b \text{fun}(t)dt$, tol= error tolerance
function fun=f(t), fun=t.∧ 2	defines $f(x) = t^2$ in an m-file
f=@(t,x) t*x-sin(t)	defines a symbolic function $f(t,x) = tx - \sin(t)$
ezplot('2*x*sin(x)', [-2,4])	plots $f(x) = 2x\sin(x)$ on $-2 \leq x \leq 4$

The remaining portion of the appendix contains the following topics:

- Scripts or m-files of programs (codes) for solving differential equations: the Euler method, modified Euler method, and the Runge–Kutta method
- Using MATLAB's build-in programs to solve differential equations
- Symbolic commands to solve differential equations
- Other common utilities and examples

B.1 Coding Algorithms for Differential Equations

Below are three programs, or codes, in MATLAB for the Euler method, the improved Euler method, and the Runge–Kutta method. The first two codes are scripts that define the right side of the differential equation by a *function handle*. They may be saved with the .m extension and run as needed. As an example, we solve a Newton's law of cooling problem

$$x' = -0.1\left(x - \left(15 + 12\cos\frac{\pi t}{12}\right)\right), \quad x(0) = 68, \quad 0 \le t \le 72.$$

The plot is shown in Figure B.1.

■ The Euler Method

```
f=@(t,x) -0.1*(x-(15+12*cos(pi*t/12)));
t0=0; x0=68; T=72; N=1000; h=(T-t0)/N; x=zeros(N+1); x(1)=x0;
for n=1:N
t(n) = t0+(n-1)*h;
x(n+1) = x(n)+h*f(t(n),x(n));
end
t=t0:h:T; plot(t,x)
```

%Labels can be added to the plot as follows:

```
xlabel('time in hours'), ylabel('temperature in deg F')
ylim([0 70]), title('Newton Law of Cooling')
```

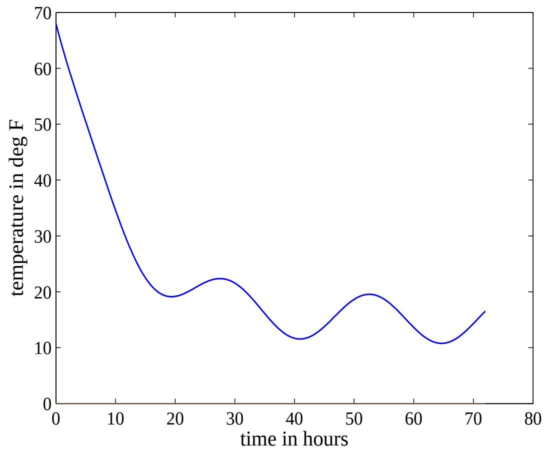

Figure B.1 The solution of $x' = -0.1(x - (15 + 12\cos(\pi t/12)))$, $x(0) = 68$, which is Newton's law of cooling. This is solved by the Euler (shown), modified Euler, and Runge–Kutta methods.

■ The Modified Euler Method

```
f=@(t,x) -0.1*(x-(15+12*cos(3.14*t/12)));
t0=0; x0=68; T=72; N=1000; h=(T-t0)/N; x=zeros(N+1);
x(1)=x0; t=t0:h:T;
for n=1:N
x(n+1) = x(n)+(h/2)*(f(t(n),x(n))+f(t(n+1),x(n)+h*f(t(n),x(n))));
end
t=t0:h:T; plot(t,x)
```

■ The Runge–Kutta Method

In this code we define the differential equation in a *function* file at end, instead of using a function handle as in the last two codes. The calling code must also be a function file, here function RungeKutta.

```
function RungeKutta
t0=0; x0=68; T=72; N=20; h=(T-t0)/N;
x=zeros(1,N+1); x(1)=x0;
for n=1:N
t(n)=t0+(n-1)*h;
k1=f(t(n),x(n)); k2=f(t(n)+0.5*h,x(n)+0.5*h*k1);
```

```
k3=f(t(n)+0.5*h,x(n)+0.5*h*k2); k4=f(t(n)+h,x(n)+h*k3);
x(n+1)=x(n)+(h/6)*(k1+2*k2+2*k3+k4);
end
time=t0:h:T; plot(time,x)
xlabel('time t'), ylabel('state x')

function dx=f(t,x)
dx=-0.1*(x-(15+12*cos(3.14*t/12)));
```

■ Two-Dimensional Runge–Kutta Method

The following m-file solves the system of two linear differential equations

$$x' = -2.9x - 4.3y, \quad y' = 0.21x - 0.78y, \quad 0 \leq t \leq 10.$$

with initial conditions $x(0) = 5$, $y(0) = 0$. N is the number of steps and h is the stepsize. Here, two figures are produced, the time series plot and a phase plane plot. The right sides of the differential equations are defined by function handles.

```
t0=0; x0=5; y0=0; T=10; N=1000; h=(T-t0)/N;
x=zeros(N+1); y=zeros(N+1);
x(1)=x0; y(1)=y0;
f=@(t,x,y) -2.9*x-4.3*y; g=@(t,x,y) 0.21*x-0.78*y;
for n=1:N
t(n)=t0+(n-1)*h;
k11=f(t(n),x(n),y(n)); k21=g(t(n),x(n),y(n));
k12=f(t(n)+0.5*h,x(n)+0.5*h*k11,y(n)+0.5*h*k21);
k22=g(t(n)+0.5*h,x(n)+0.5*h*k11,y(n)+0.5*h*k21);
k13=f(t(n)+0.5*h,x(n)+0.5*h*k12,y(n)+0.5*h*k22);
k23=g(t(n)+0.5*h,x(n)+0.5*h*k12,y(n)+0.5*h*k22);
k14=f(t(n),x(n)+h*k13,y(n)+h*k23);
k24=g(t(n),x(n)+h*k13,y(n)+h*k23);
x(n+1)=x(n)+(h/6)*(k11+2*k12+2*k13+k14);
y(n+1)=y(n)+(h/6)*(k21+2*k22+2*k23+k24);
end
figure(1)
tau=t0:h:T; plot(tau,x,'r',tau,y,'g',tau,0)
xlabel('time'), ylabel('x, y'), title('Time Series')
legend('x','y')
figure(2)
tau=t0:h:T; plot(x,y), xlabel('x'), ylabel('y'), title('Phase Plane')
```

B.2 MATLAB's Built-in ODE Solvers

MATLAB contains several built-in solvers that numerically compute the solution to an initial value problem. To use these routines we can define both the differential equation and the calling routine in *function* files, or we can define the differential equation *symbolically* using a function handle. The files below use the package ode45, which is a Runge–Kutta type solver.

■ Single Equation

Consider the initial value problem

$$x' = 2x(1 - 0.3x) + \cos 4t, \quad 0 < t < 3, \ x(0) = 0.1.$$

```
%The calling file is defined as follows:
function diffeq
trange = [0  3]; ic=0.1;
[t, x] =ode45(@dx,trange,ic);
plot(t,x,'*–')
%The differential equation is defined as follows:
function dx = f(t,x)
dx = 2*x.*(1-0.3*x)+cos(4*t);
```

■ Second-Order Equations and Systems

Consider the famous **van der Pol equation**

$$x'' + a(x^2 - 1)x' + x = 0,$$

where a is a positive parameter. We convert it to a system in the usual way:

$$x' = y, \quad y' = a(1 - x^2)y - x.$$

Take initial conditions $x(0) = 2$, $y(0) = 0$. Two plots, the time series (Figure B.2) and the phase plane (Figure B.3), are shown. The reader is urged to run the same program for other values of a, say, $a = 1000$. In this latter case the equation is *stiff*, meaning there are very rapid changes in x and y over a small time interval. A Runge–Kutta method may be inadequate for this case, and we can use a stiff solver, one of which in MATLAB is ODE15s.

```
function vanderpoleqn
y0=[2;0]; tspan=[0,20]; a=1;
[t,y]=ode45(@vanderpol, tspan, y0);
figure(1)
plot(t,y(:,1)), xlabel('t'), ylabel('solution y')
```

```
title('van der Pol Equation, a=1')
figure(2)
plot(y(:,1),y(:,2)), xlabel('x'), ylabel('y')
function dy=vanderpol(t,y)
a=1;
dy=[y(2);a*(1-y(1)*y(1))*y(2)-y(1)];
```

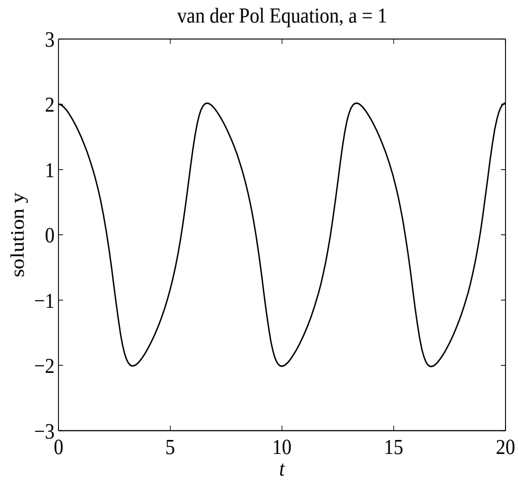

Figure B.2 The solution $x = x(t)$ of the second-order the van der Pol equation when $a = 1$.

■ The Duffing Equation

The **Duffing equation**

$$x'' + \gamma x' + kx + \beta x^3 = A\cos\omega t,$$

an important and famous nonlinear, forced oscillator equation. γ is the damping constant, k the stiffness, β the nonlinear effect, A the amplitude, and ω the forcing frequency. It has extraordinary properties, depending on the values of the parameters.[1] We use the solver ode23s, which is a *stiff* solver that can adapt to rapid changes in x over short time intervals. We take $\gamma = k = \beta = 1$ and $A = 0.8$, $\omega = 3$. See Figure B.4.

[1] For example, see the *Wikipedia* article at http://en.wikipedia.org/wiki/Duffing equation.

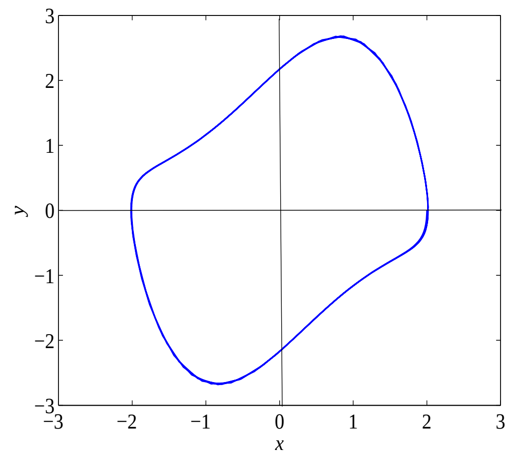

Figure B.3 The corresponding phase plane orbit, which is periodic, for the van der Pol equation when $a = 1$. It can be shown that that this solution is a limit cycle.

```
function DuffingEq
[T,X]=ode23s(@f,[0 1000],[1 0]);
subplot(1,2,1), plot(X(:,1),X(:,2))
subplot(1,2,2), plot(T,X(:,1))
function dx=f(t,x)
dx=[x(2); -x(2)-x(1)-x(1)^3+0.8*cos(3*t)];
```

B.3 Symbolic Solutions Using **dsolve**

- The following script *symbolically* solves the logistic equation $y' = ry(1 - y/K)$, $y(0) = y_0$ and plots the solution using dsolve. The command vectorize turns a symbolic solution into a vector solution that MATLAB can evaluate and plot. (The plot is not shown.)

```
y=dsolve('Dy=r*y*(1-(1/K)*y)','y(0)=y0');
y=vectorize(y);
r=0.5; K=150; y0=15; t=0:.05:20; y=eval(y);
plot(t,y), ylim([0 K+10]), title('Logistic Growth')
xlabel('time (years)'), ylabel('Population')
```

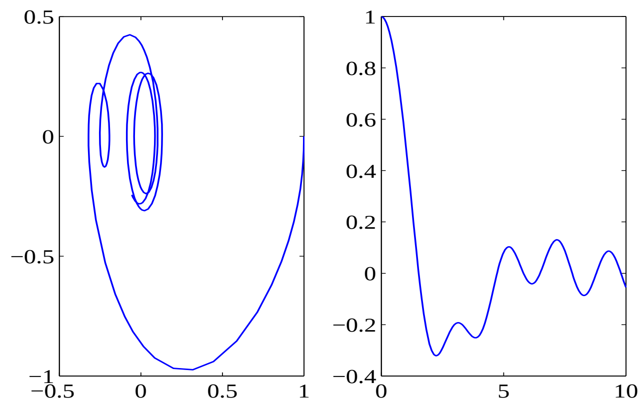

Figure B.4 Numerical solution of the Duffing equation: (Left) The orbit in the xy phase plane, and (Right) the time series $x = x(t)$. The initial condition is $x(0) = 1$, $y(0) = 0$.

- Solve and plot a symbolic solution of $x' = tx^2, \; x(-2) = 1$ using dsolve and ezplot.

```
sol = dsolve('Dx=t*x∧2','x(-2)=1','t');
ezplot(sol,[-2,2])
```

- Next we solve and plot the solution to an initial value problem,

$$x' = tx^2, \; x(-2) = 1, \;\; -2 \leq t \leq 2,$$

using ode45 and a function handle:

```
f=@(t,x) t*x∧2;
[T,X]=ode45(f,[-2,2],1);
plot(T,X)

x=dsolve('25*D2x+10*Dx+226*x=901*cos(3*t)','x(0)=0, Dx(0)=0');
x=simple(x)
ezplot(x,[0,6*pi])
```

- Solve the system, $x' = rx + 4y, \;\; y' = 4x - 3y$, using dsolve:

```
[x,y] = dsolve('Dx=r*x+4*y, Dy =4*x-3*y', 'x(0) = a, y(0) = b');
x=vectorize(x), y=vectorize(y);
```

```
a=1; b=3; r=1; t=1:0.01:2;
x=eval(x); y=eval(y);
plot(t,x,t,y)
```

B.4 Other Routines

■ Plotting Antiderivatives

Calculate and plot the antiderivative, or function with a variable upper limit:

$$u(t) = \int_0^t e^{-x^2} dx.$$

```
x=0:0.1:5; u=exp(-x.∧2);
vals=cumtrapz(x,u);
t=x; plot(t,vals)
xlabel('time '), ylabel('u(t)')
```

■ Slope and Direction Field

To plot the slope field (Figure B.5) of the equation

$$x' = t + \sin x,$$

in the region $-3 \le t \le 3$, $-2 \le x \le 2$, use the commands:

```
[t,x]=meshgrid(-3:.3:3,-2:.3:2);
dx = t+sin(x); dt = ones(size(dx));
dxu = dx./sqrt(dt.∧2+dx.∧2);
dtu = dt./sqrt(dt.∧2+dx.∧2);
quiver(t,x,dtu,dxu)
```

To plot the direction field of the system

$$x' = x(8 - 4x - y), \quad y' = y(3 - 3x - y)$$

in the region $0 < x < 3$, $0 < y < 4$, we again use the quiver command. Direction fields have both direction and length, but usually we are interested only in directions; therefore, each vector is divided by its length (making it unit length), and then multiplying by 0.5. The figure is not shown.

```
[x,y] = meshgrid(0:0.3:3, 0:0.4:4);
dx = x.*(8-4*x-y); dy = y.*(3-3*x-y);
```

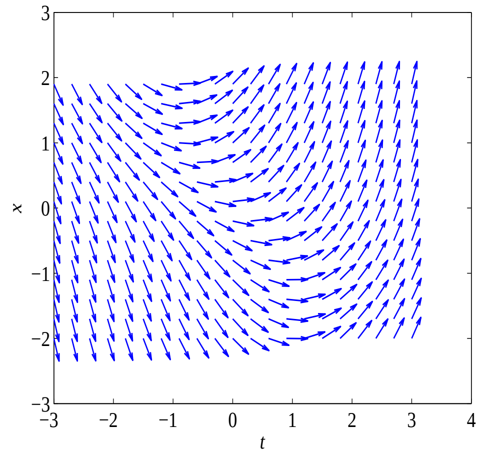

Figure B.5 Slope field diagram for the equation $x' = t + \sin x$.

```
dux=dx./sqrt(dx.∧2+dy.∧2); duy=dy./sqrt(dx.∧2+dy.∧2);
quiver(x,y,dux,duy,0.5)
```

■ Roots of a Polynomial

Solving higher order linear equations requires finding roots of the characteristic equation. If the characteristic equation is a cubic equation $a\lambda^3+b\lambda^2+c\lambda+d=0$, then the roots are found by the command roots([a b c d]). For example, if $\lambda^3 - 3\lambda^2 + 1 = 0$ then roots([1 -3 0 1]) returns the roots -0.53, 0.65, 2.88.

B.5 Examples

- **Plotting Commands.** Define a function $f(t) = \cos(t)t$ by: f = @(t) cos(t)*t.

  To evaluate the f at given t-value: f(0.5).

  To plot the graph of f over an interval: ezplot(f,[0,20])

  To find a zero of the function near an initial guess: fzero(f, 2)

  To define a function of several variables: F = @(x,y) x∧4 + y∧4

  To evaluate the function for given values of x, y: F(1,2)

To graph $F(x, y)$ over a rectangle in the x, y plane: ezsurf(F,[-2,2,-2,2])

- **Numerical Solution**. Solve the initial value problem $x' = tx^2$, $x(-2) = 1$, and plot the solution over the interval $[-2, 2]$.

```
f = @(t,x) t*x∧2
[ts,xs] = ode45(f,[-2,2],1);
plot(ts,xs)
```

To plot *several solution curves* use the commands hold on and hold off. After obtaining the first plot type hold on, then all subsequent commands plot in the same window. After the last plot command type hold off. For example (see Figure B.6), plot the direction field and the 13 solution curves with the initial conditions $x(-2) = -0.4, -0.2, \ldots, 1.8, 2$:

```
for x0=-0.4:0.2:2
[ts,xs] = ode45(f,[-2,2],x0); plot(ts,xs), hold on
end
hold off
```

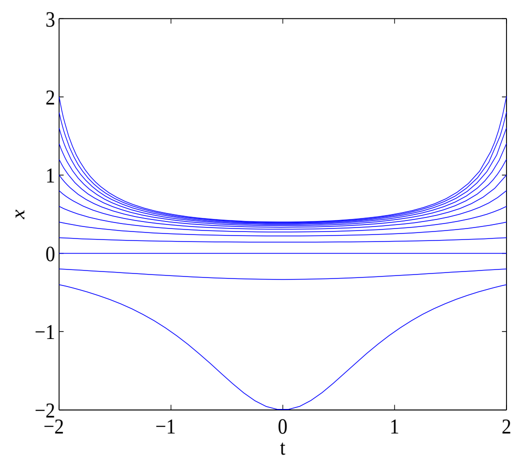

Figure B.6 Plots of several integral curves of the differential equation $x' = tx^2$.

- **Symbolic Solution.** To solve a differential equation symbolically enter

sol = dsolve('Dx=t*x∧2','t').

When MATLAB cannot find a solution it returns an empty symbol. If it finds several solutions it returns a vector of solutions. Here there are two solutions and MATLAB returns a vector sol with two components: sol(1) is 0 and sol(2) is -1/(t∧2/2 + C1) with an arbitrary constant C1. You can substitute values for the constant using subs(sol,'C1',value). For example, to set C1 in sol(2) to 5 use subs(sol(2),'C1',5). You can simplify the solution using simple(sol).

To solve an initial value problem specify an initial condition:

```
sol = dsolve('Dx=t*x∧2','x(-2)=-4','t')
```

To plot the solution use ezplot(sol,[t0,t1]). For example (Figure B.7):

```
sol = dsolve('Dx=t*x∧2','x(-2)=-0.4','t')
ezplot( sol , [-2 2])
```

To obtain numerical values at one or more t values use subs(sol,'t',tval).

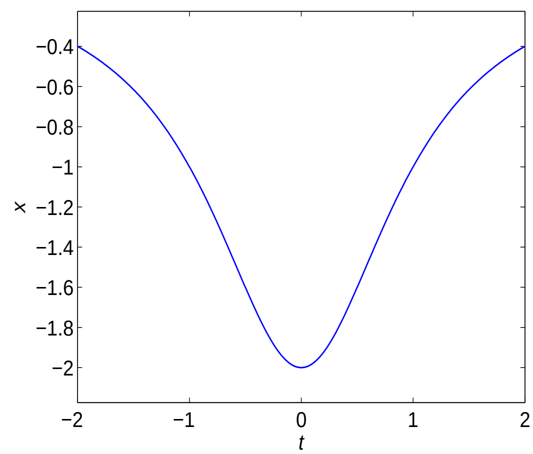

Figure B.7 The solution to $x' = tx^2,\ x(-2) = -0.4$

- **Solution of a system.** Consider the system

$$y1' = y2,$$
$$y2' = -\sin(y1) + \sin(5t)$$

with the initial conditions $y1(0) = 1,\ y2(0) = 0$, which models a forced pendulum. Here $y1$ and $y2$ are the states, instead of the usual x and y because of ease of coding in MATLAB.

First we define the right sides of the differential equation as a vector function (handle),

```
f = @(t,y) [y(2); -sin(y(1))+sin(5*t)]
```

To plot the solution over the range $0 \leq t \leq 20$ with initial values $[1;0]$ we enter

```
[ts,ys] = ode45(f,[0,20],[1;0]); plot(ts,ys)
```

Type `plot(ts,ys(:,1))` to plot only $y1(t)$.

To plot orbits in the phase plane (Figure B.8) enter

```
plot(ys(:,1),ys(:,2))
```

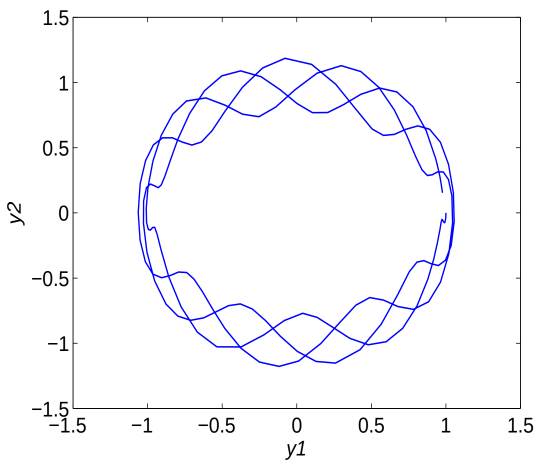

Figure B.8 Phase plane plot of the solution to $y1' = y2,\ y2' = -\sin(y1) + \sin(5t),\ y1(0) = 1,\ y2(0) = 0$.

- **Using dsolve.** To obtain a symbolic solutions we use the `dsolve` command. For the second-order equation

$$y'' = -y + \sin(5t),$$

enter

```
sol = dsolve('D2y = -y + sin(5*t)','t')
```

If you want to simplify the answer enter

```
s = simple(sol)
```

To solve the equation with initial conditions $y(0) = 1,\ y'(0) = 0$ use

```
sol = dsolve('D2y = -y + sin(5*t)','y(0)=1','Dy(0)=0','t')
```

Then use s = simple(sol) to simplify, if needed. To plot the solution curve (see Figure B.9) use ezplot:

```
ezplot(sol, [0,20])
```

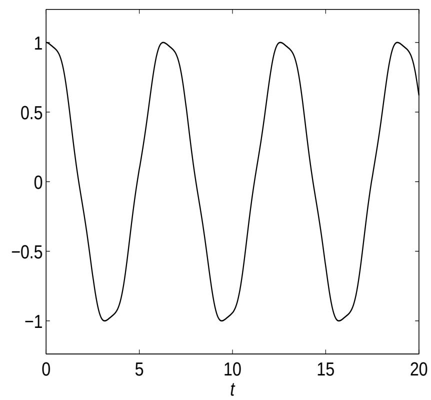

Figure B.9 solution to $y'' = -y + \sin t,\ y(0) = 1,\ y'(0) = 0.$

The dsolve command gives the output

```
sol =
y1: [1x1 sym]
y2: [1x1 sym]
```

which means that the two components of the solution can be found by entering sol.y1 and sol.y2. We can always simplify using, for example, s2 = simple(sol.y2).

To plot the solution curves use ezplot:

```
ezplot(sol.y1, [0,20])
hold on
ezplot(sol.y2, [0,20])
hold off
```

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Index