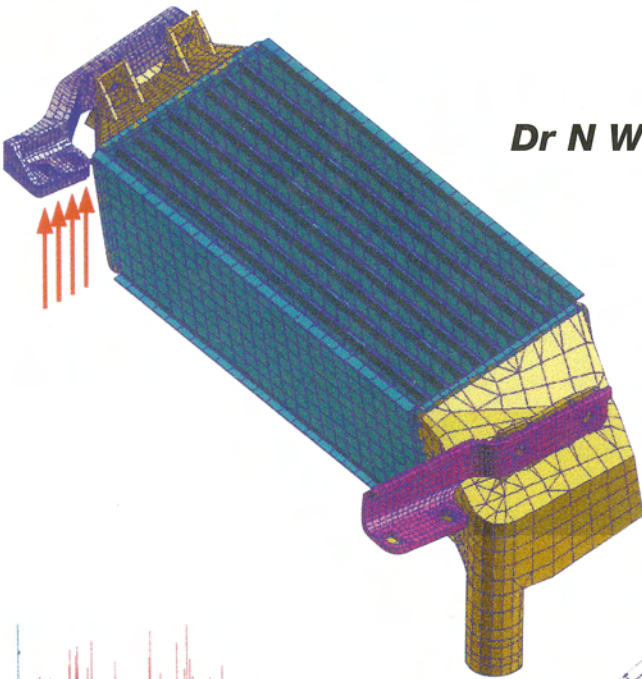


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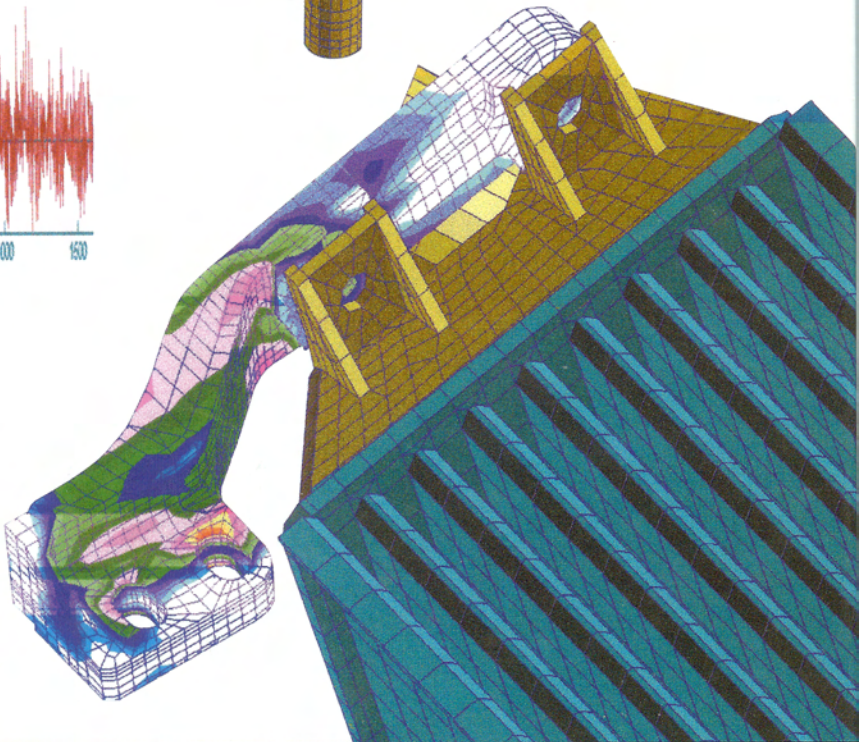
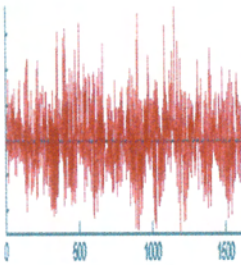
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*Dr N W M Bishop*

*&*

*Dr F Sherratt*



## *Finite Element Based Fatigue Calculations*

# Finite Element Based Fatigue Calculations

**Authors:**

**Dr NWM Bishop<sup>†</sup> & Dr F Sherratt<sup>‡</sup>**

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Finally, my co-author and I would like to dedicate the book to the memory of Ian Austen who will always be remembered as a friend.

Dr NWM Bishop  
Farnham, UK, July 2000

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# **PREFACE**

To be inserted by John Smart

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# 1 Introduction

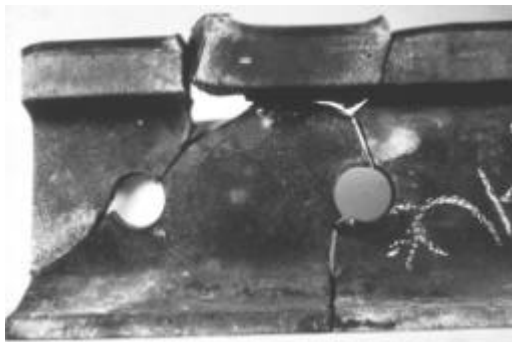
Fatigue analysis procedures for the design of modern structures rely on techniques, which have been developed over the last 100 years or so. Initially these techniques were relatively simple procedures, which compared measured constant amplitude stresses (from prototype tests) with material data from test coupons. These techniques have become progressively more sophisticated with the introduction of strain based techniques to deal with local plasticity effects. Nowadays, variable amplitude stress responses can be dealt with. Furthermore, techniques exist to predict how fast a crack will grow through a component, instead of the more limited capability to simply predict the time to failure. Even more recently techniques have been introduced to deal with the occurrence of stresses in more than one principal direction (multi-axial fatigue) and to deal with vibrating structures where responses are predicted as PSDs (Power Spectral Densities) of stress.

The process of crack growth, the basis of all so-called fatigue damage, is actually a very complicated phenomenon. Before 1900 it had already been recognised that cyclic loading could result in progressive failure at stress, or strain, levels well below those that would be expected to cause failure on a single application of load. All sorts of explanations have since been put forward but we now know that, in general terms, cracks 'initiate' in the form of "to and fro" slip on crystals along the direction of maximum shear. These slip bands turn into comparatively wide bands and then tend to merge into cracks that then grow perpendicular to the applied strain. In the early stages of crack formation the cracks tend to be influenced by a variety of microscopic effects as the crack tip grows through such things as grain boundaries and other crystallographic details.

In this way the concept of different stages of crack growth has been observed. Attempts have been made to define so-called short cracks that govern this early behaviour. And, it is reasonable to say that the vast majority of fatigue research has concentrated on this area. It is impossible to find full agreement on what is meant by the term short crack but in a steel this might be all cracks less than 1-2mm. Figure 1 shows a post fatigue failure surface where, at the point of failure, the crack was several orders of magnitude larger than this. Figure 2 shows a failure crack that occurred for a railway component.



*Figure 1. A fatigue failure surface*



*Figure 2. Failure of a railway track component*

The frightening aspect of fatigue failure mechanisms is that initial crack growth can often go undetected. It is not until a critical component becomes inoperative that the result of the fatigue failure becomes apparent (see Figure 3 and Figure 4).

In design, three core fatigue methodologies have become established. The first two techniques that were developed do not model the crack growth process at all. Instead, they use the concept of similitude to determine the number of cycles to failure; ‘Failure’ being defined as some predetermined crack length, or loss of stiffness, or separation of the component being designed. This means that the relationship between component life and load level in a test specimen can be compared directly with that expected in service (assuming the component is tested under identical conditions). The first of these two methods is based on stress, the so-called ***Stress-Life*** (S-N, nominal stress, or total life) method. The second, and more recent technique is based on strain, the so-called ***Strain-Life*** (Local-Stress-Strain, Crack-Initiation, Manson-Coffin or Critical-Location Approach - CLA) method.



*Figure 3. Crane failures resulting from crack growth*



*Figure 4. Railway accidents can be very destructive*

The third, and most recently developed method, deals with **Crack-Propagation** and relies on the observation that once cracks become established they have a stable growth period. This is usually described using linear elastic fracture mechanics. It further relies on the assumption that crack growth rates are proportional to the applied stress intensity (a function of crack length, geometry and stress level).

Today, 95% of all fatigue design calculations are covered by one of these three approaches, i.e., Stress-Life, Strain-Life or Crack-Propagation. Furthermore, since stress, or strain, are the governing variables it has been usual to test prototype components in order to obtain the required data needed for the fatigue analysis. However, with the introduction of Finite Element Analysis (FEA) techniques, has come the possibility of doing fatigue calculations long before a prototype exists. Furthermore, a dramatic improvement in computing power has made FE based fatigue life calculations a routine task. FE has been around for some time and is a now a mature technology. The purpose of this book is to provide an introduction to the basic underlying concepts of fatigue analysis within the FE environment. This goal can be stated further as to give engineers involved in FE a basic understanding of fatigue; and to give engineers involved in fatigue a basic understanding of FE.



## 2 Why FEA Based Fatigue Design?

When referring to loaded components fatigue may be defined as

*Failure under a repeated or otherwise varying load which never reaches a level sufficient to cause failure in a single application.*

Fatigue usually involves the initiation and growth of a crack until it reaches a critical size, sometimes causing separation into two or more parts. Other criteria of deterioration apply in some circumstances, but this covers components based on iron, steel or aluminium which are by far the most common materials. In these crystalline metals the initiation process is caused by slip on crystal planes, controlled by time-varying shear stresses. The start of a fatigue failure is therefore a strictly local process and it is also one that depends on the dynamics of the system. The time history of stress or strain, at the exact location where a crack is going to start, is the critical factor and the general distribution of these parameters throughout the component is of secondary interest. This is precisely why Finite Element Analysis is important in this discipline. By using FEA an analyst can choose any location within a model and concentrate attention on it, using the intrinsic ability of the technique to bring in dynamic effects.

Most fatigue analysis has concentrated on the metals named earlier. The non-linear behaviour shown by elastomers and polymers, at quite low stresses, has made them more difficult subjects, and the application of FEA to these materials is more a research topic than a routine design exercise. Thermo-mechanical effects also cause some difficulty. Currently, the role of the main commercial FE fatigue codes is to enable the stresses derived from a thermal analysis to be used independently of temperature to give a fatigue life. Effects which can not easily be incorporated are “thermal creep” (this will cause crack extension), “changing material properties” (these will change with temperature) and “oxidation/corrosion” (may enhance or retard crack growth). These topics are usually the subject of research programs. Thermo-mechanical effects are included in some ongoing software development programs.

### 2.1 The Elements of a Life Estimation System.

For many years the fatigue analysis process has been thought of as following the logic set out in *Figure 5*. In this overview the three input parameters, geometry, materials and loading, are regarded as having similar functions. In practice most

analysis has followed the model shown in Figure 6. The geometry and loading are initially used together to produce a stress-time ( $\sigma - t$ ) or strain-time ( $\epsilon - t$ ) history at a point likely to be critical. Material fatigue properties are then introduced to estimate life. The only material properties needed in the first step are things like Young's Modulus, the elastic-plastic stress-strain curve, etc., which are not true fatigue properties.

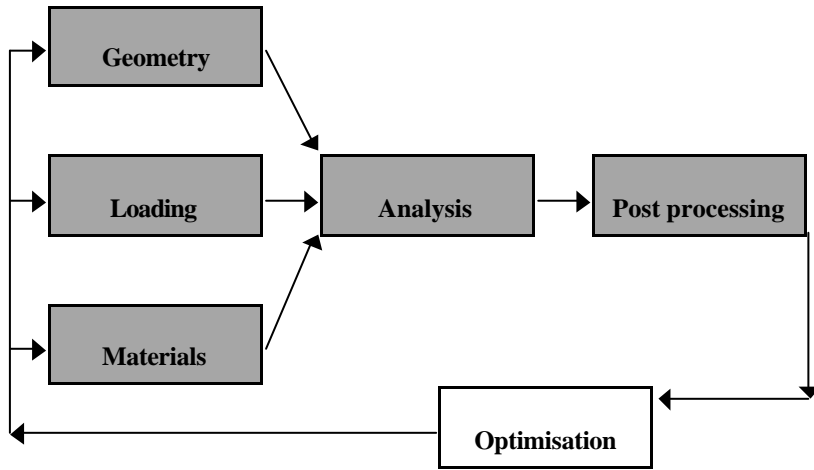


Figure 5. A conventional view of the fatigue analysis process

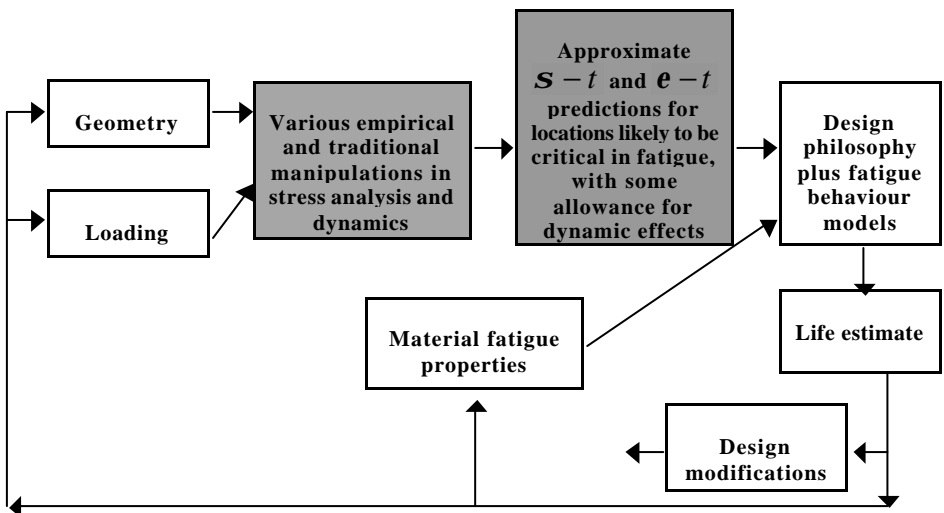


Figure 6. An alternative view of traditional life estimation procedures

The step from overall geometry and generalised loading, to a detailed map of local stress and strain in a component has traditionally required the use of a variety of techniques, some with a sound analytical background but many with a simple empirical basis. If the loading is fluctuating with time, as it always will be in the case of fatigue, a further set of uncertainties enters. Using FEA gives tighter control over the move from general geometry and loading to local parameters, and allows dynamic factors to be dealt with more analytically. A model like Figure 7 then describes the process, emphasising the importance of FEA in a situation where analysis at precise locations is essential.

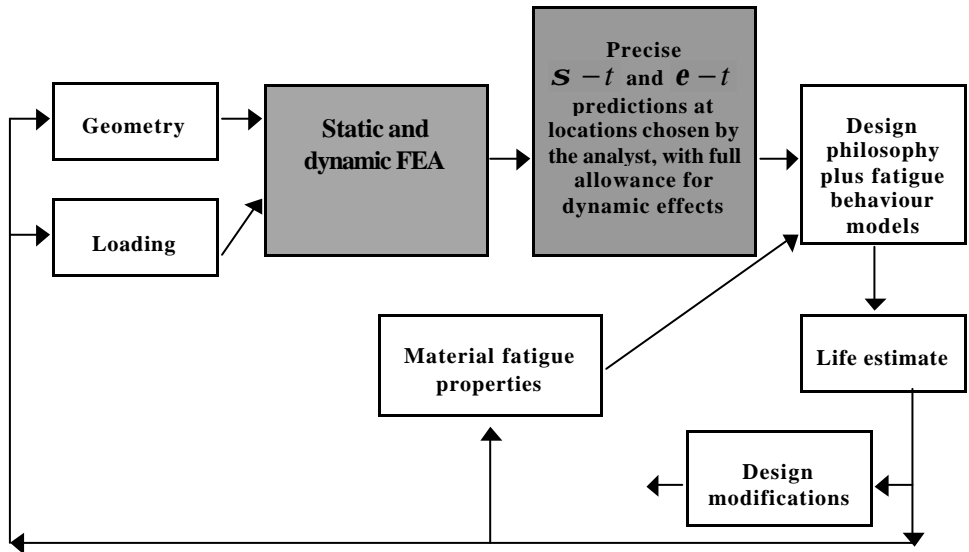


Figure 7. A life estimation scheme using FEA

The role of a fatigue calculation varies according to the component, the loading and the situation in which the component is to be used. A component with simple geometry and simple loading which is to be used in a situation where failure would cause only minor inconvenience may be manufactured and put into service purely on the basis of a calculation. This is particularly so if only small numbers of the article are to be made. If the situation is more complex, and the penalties for wrong estimates are higher, verification of the calculation by testing becomes necessary. A simple test may still be good enough, but in this case FEA is unlikely to be in use.

A more typical situation is one in which a component with complicated geometry and multiple loads, is to be produced in large numbers, require minimum weight and be used in a safety-critical application. Prototype components or full-scale assemblies then need to be tested under loading as close as possible to that



expected in service. This is a very expensive operation. In addition to expense, a significant drawback with this type of testing is that it cannot be undertaken until a prototype exists. If a design problem then occurs it is likely to be difficult and expensive to rectify. The more accurate and reliable the life prediction process becomes the less likely it is that late modifications will be needed. The main contribution of FEA based fatigue tools is then to enable reliable fatigue life calculations to be done at the design stage of a development process, long before tests are possible

## 2.2 An Overview of the FEA Based Fatigue Environment

Figure 8 gives an overview of the FEA based fatigue environment. The three plots on the left indicate the FEA results, applied loading and materials information. The three plots on the right show the types of result visualisation that are possible. The centre box indicates the types of fatigue calculations that can be done.

All of the fatigue techniques specified in Figure 8 are completely, or substantially, based on one of the three standard life estimation methods, i.e., Stress-Life, Strain-Life or Crack-Propagation, described in detail later.

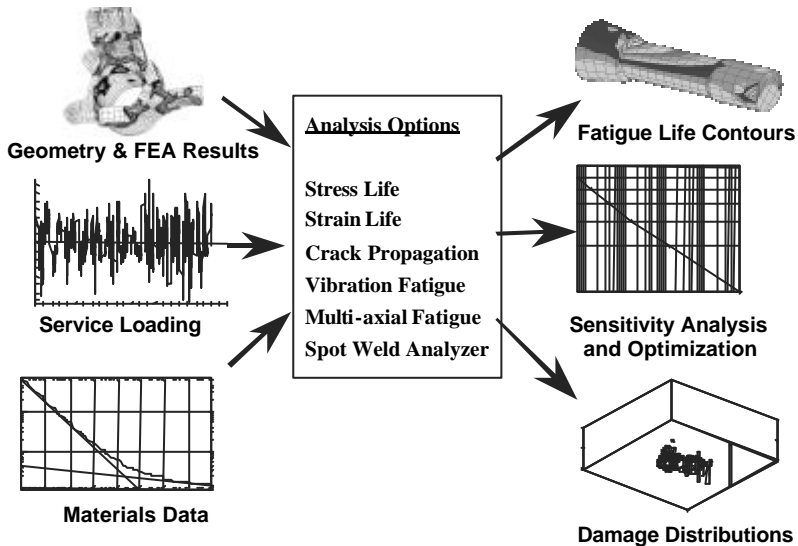


Figure 8. An overview of the FEA based fatigue environment

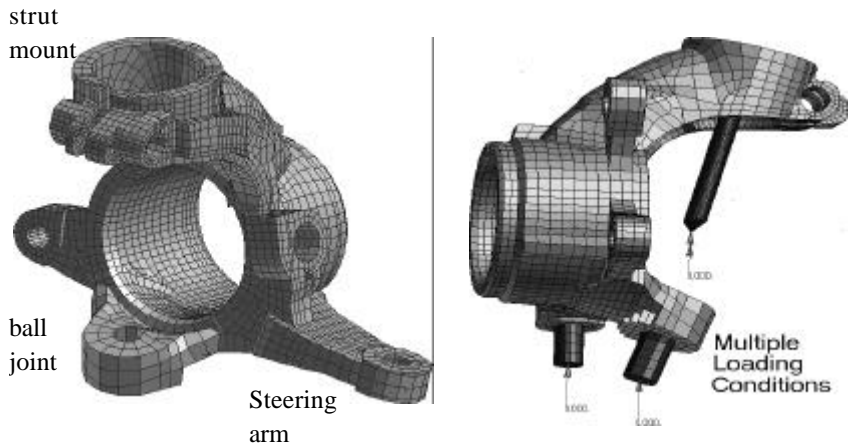
To gain a general picture of the steps needed, consider the model shown in Figure 9. This is a steering knuckle subjected to a complex, multiple load-case loading environment. The component is a steering knuckle from a car. It is cast from a

spheroidal graphite cast iron. The obvious features are the strut mount at the top, the lower ball joint at the bottom and the steering arm on the right. The wheel spindle goes through the large cylindrical hole in the central part.

When the vehicle is driven through a cobblestone slalom, loads are applied to the component via the strut mount, the lower ball joint, the steering tie rod and the wheel axis.

In the FE analysis the loads are applied via loading devices in an attempt to make the transfer of loads to the component as realistic as possible. This is done using devices made from elements rather than MPCs (multi point constraints).

The model is constrained at the wheel centre (again through element loading devices) and 12 load cases are applied: 3 forces (1000N in x-y-z) at the lower ball joint, the steering arm and the strut mount, and 3 moments (1000 Nmm) at the strut mount. Three of the force time histories (torque, horizontal and vertical forces applied to 1 of the 4 load application points on the model) are shown in Figure 10. A linear combination of these 12 load cases can describe any loading condition that occurs during the test track event.



*Figure 9. A Car steering knuckle*

The stress distribution caused by a unit load applied to 1 of these 12 points is shown in Figure 11.

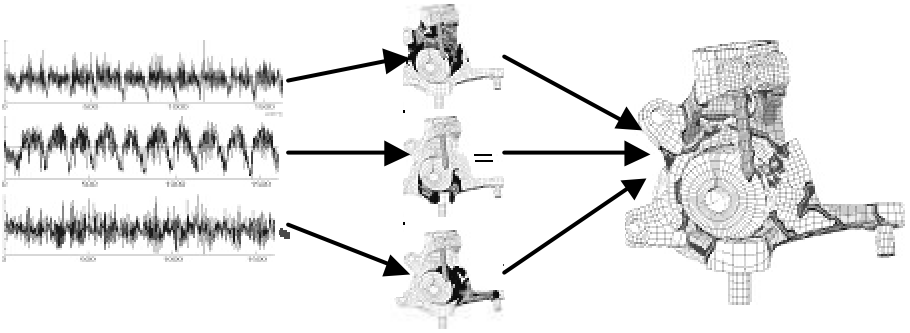


Figure 10. Three of the 12 loadings applied to the model

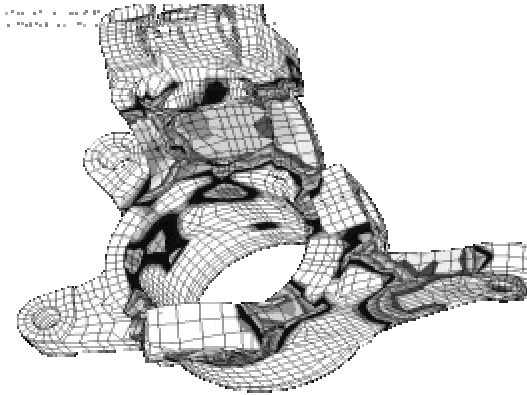
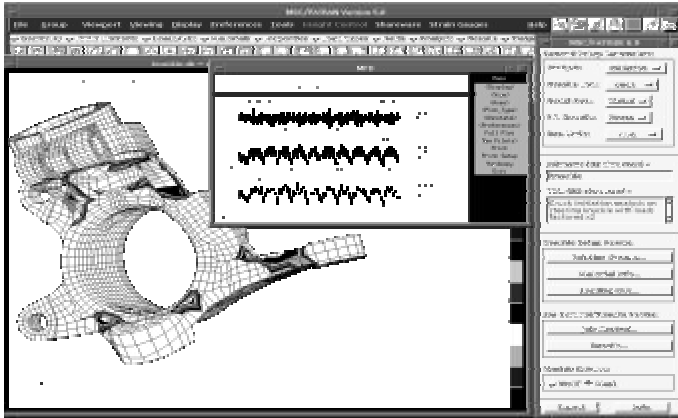


Figure 11. Static stress distribution for a point load at 1 load application point.

Most FEA based fatigue software packages allow the effects of all 12 force time histories to be scaled and superimposed. One resultant stress parameter (e.g., maximum principal stress, Von Mises stress, Tresca stress, etc) is then used to determine a fatigue life result for each point on the model. The linear superposition is accomplished with the following equation,

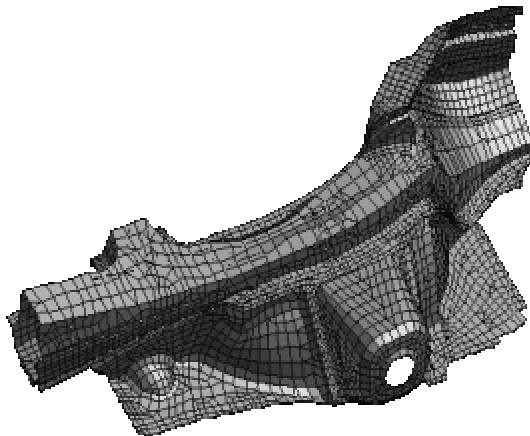
$$\mathbf{s}_{ij}(t) = \sum_k P_k(t) \left( \frac{\mathbf{s}_{ij,k}}{P_{k,fea}} \right)$$

where  $P_k(t)$  is the force time history,  $P_{k,fea}$  is the magnitude of the force used to produce the static load case (usually unity) and  $\mathbf{s}_{ij,k}$  is the static stress result at point ij, for load case k. Figure 12 shows an example of the fatigue life result (plotted using the FE based fatigue package MSC.Fatigue, developed by MSC Software).



*Figure 12 . The fatigue damage caused by the application of 12 force time histories.*

In Figure 8 the types of computation indicated in the ‘analysis’ box include the Stress-Life, Strain-Life and Crack-Growth methodologies. In addition, a number of other specialised approaches are mentioned which are based fully, or in part, on these approaches. For instance, a spot weld analysis can be undertaken using the Stress-Life approach, the only difference being that the stresses have to be derived in the spot weld nuggets, a task that can be accomplished using FEA methods. Figure 13 and Figure 14 show an FEA model and fatigue life results for just such a task. Vibration fatigue and multi-axial fatigue concepts are described more fully later.



*Figure 13. An FEA model of a spot weld assembly*

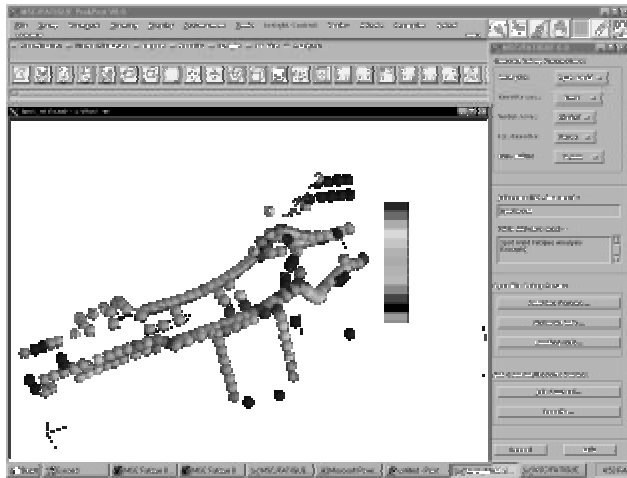


Figure 14. Fatigue life results for each spot weld. The life is indicated here by grey shading but the spheres can also be sized according to their life

### 2.3 FE Hints and Tips – Why FEA

It is very important to appreciate the issue of accuracy when performing fatigue life calculations with FE models. Small changes in how structural behaviour is modelled, as well as detailed modelling procedures such as meshing, can have quite large effects on predicted stresses. A common and understandable assertion is, therefore, that FE based calculations should only be undertaken when correlated against test. Actually, even here there are problems because if the same test is undertaken twice two different results will be obtained. Another common statement concerning fatigue results from test, or FE, is that if the results are within a factor of 2 they are the same result.

This discussion leads us to the view that that engineers doing fatigue calculations must have an appreciation of the dimensions involved in the various parts of the behaviour. For example, if an external load doubles what happens to fatigue life in critical regions? This, of course, depends on the external load type (force, displacement, acceleration etc). But assuming that a linear relationship exists and stresses also double what happens to fatigue life? In order to understand this it is important to have an appreciation of material fatigue curves. For example, a typical aluminium may have a fatigue material slope specified as  $b = 10$  (see later section 4.2) in which case a doubling of applied load would cause a change in fatigue life or damage of  $2^{10}$  or approximately 1000.

Let us now reconsider the issue of accuracy that we raised above. Behind any FE based fatigue life calculation there are three underlying effects on fatigue life.

Firstly, small changes in modelling practice can cause big effects on fatigue life. Secondly, small changes in applied loading can cause big changes in fatigue life. And thirdly, in tests, components subjected to the same loading can experience large variations in fatigue life. So, one might assume that there is little point in the whole process. This is definitely not the case. But it is important to recognise that,

*Absolute fatigue life is unobtainable*

However, it should also be recognised that the results do provide a single metric of acceptance for a complete customer route or other sign off criterion. They also allow A to B comparisons to be made without the need for complete accuracy. Also, such analyses allow robustness studies through sensitivity analyses to be achieved. Optimisation studies can also be performed as well as comparisons between different loading scenarios. These are the underlying benefits of FE based fatigue calculations.



## 3 Different Philosophies and Life Estimation Models.

Intelligent use of FEA software for fatigue design needs some knowledge of the principles of life estimation. Detailed knowledge is not strictly necessary, since this will have been incorporated into the software, and is in any case available in texts like Bannantine et.al (1990). Figure 7 above shows that analysts using the specialist software must still make decisions, and this chapter will lay out the broad basis of those choices. Later chapters will give further details for users who need this.

### 3.1 Design Philosophies

Before attempting to carry out a fatigue calculation, or even choosing a way of doing that calculation, it is necessary in critical situations to decide on a design philosophy. Is the component to be regularly inspected, for instance, and taken out of service when cracks are found? Or, is it to be taken out of service before there is any possibility of cracks forming. These considerations apply whether FEA is to be used or not.

The three main approaches are Safe-Life, Fail-Safe and Damage-Tolerant. To illustrate the three consider the design of a stool.

#### 3.1.1 Safe-Life.

In the *Safe-Life* philosophy products are designed to survive a specific design life with a chosen reserve. Calculation alone may be used, or there may be some testing. The design life will then be some fraction of the estimated life, typically about one fifth for safety-critical components. In general, this philosophy results in somewhat optimised structures such as a stool with three legs (see Figure 15), which will fall over if one breaks. The penalty is that components have to be taken out of service when it is likely that they still have substantial remaining life. Furthermore, with the approach there is always a possibility that components will be very reliable and over designed.



### 3.1.2 Fail-Safe.

To reduce some of this waste of useful fatigue life, and maintain or improve the operating safety of a component in the later stages of its life, the philosophy of **Fail-Safe** may be adopted. A stool having six legs illustrates this (see Figure 16). If one leg were to break off the stool would remain standing until repairs could be made. An inspection procedure to detect the failure is needed, and a clear definition of action to be taken following this inspection must be specified.

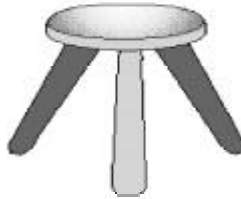


Figure 15. A 'Safe-Life' stool. Any less than 3 legs and it would fall over!

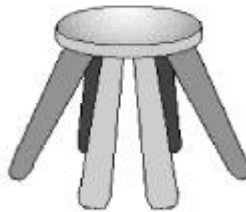


Figure 16. A 'Fail-Safe' stool. Failure of one leg would not result in overall failure.

### 3.1.3 Damage-Tolerant.

In illustrating the Fail-Safe approach it was assumed that complete separation of one of the legs was the “failure” event. A more subtle inspection criterion is to inspect all legs periodically to see whether or not cracks have started. If a crack is found the stool could be taken out of service immediately, but it is possible that failure is not imminent. We would then need information about the loading, how that loading affected crack growth and how big a crack would be needed to cause collapse. This is the basis of **Damage-Tolerant** design (see Figure 17). It needs a close understanding of how cracks grow steadily under varying load and extend catastrophically when they reach a certain length. This is the discipline of Fracture Mechanics, covered in a later chapter. Note that when using this approach a clearly defined inspection procedure and agreed action following the result of this inspection is required, as it is with the Fail-Safe approach.

Note that structural redundancy is not strictly needed. A stool with three legs would still be acceptable if the legs were made of wood, which tolerates cracks.



*Figure 17. A 'Damage-Tolerant' stool. The stool has some built in redundancy and is inspected regularly*

### **3.1.4 Integrated Durability Management**

In the design and development of a large system such as a complete aircraft more than one of the above approaches may be used in different parts of the system. Co-ordination of effort is then needed. The overall process is often called Integrated Durability Management. This implies that design, testing and production are co-ordinated to ensure that products are developed to meet the required life within cost and on time. In modern conditions FEA can often ensure that coherent approaches are used in the various areas.

## **3.2 Life Prediction Methods**

Once a local stress-time or strain-time history has been established for a point likely to be critical a fatigue analysis method must be chosen. Most FEA based fatigue packages have three main life prediction methods and it is sometimes useful to relate these to the stages of fatigue illustrated in Figure 18. According to this view, total life is made up of a crack initiation phase and a crack propagation phase. The proportion which each contributes will vary with the geometry, the loading and especially with the material. For example, ductile steel has a large proportion of life in the propagation stage, but brittle ceramics or cast iron have a much shorter life in the propagation phase. In addition the definition of total life will vary for different components. Complete separation into two pieces is usually assumed, but the presence of a crack of a defined length may be the criterion.

Although the above concept defines a useful visualisation tool, it has little physical justification. Physicists will point out that the Strain-Life approach does not, in fact, usually identify the initiation of a crack. Instead it identifies the time taken for

a crack to grow from an undetectable size to one that can be found using available inspection tools.

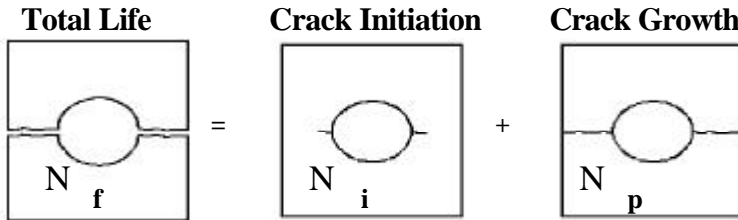


Figure 18. An idealisation of the fatigue design process

The three main analysis methods available are Stress-Life, Strain-Life and Crack-Propagation. Enlarging on this:

### 3.2.1 Stress-Life (S-N, or Nominal Stress) Approach.

This is normally used for total life calculation. A stress-time history is estimated for some part of the component judged to be representative. Factors derived from geometry are then applied to turn this into a stress-time history at the point where a crack is likely to start. Traditionally these have been taken from tables and graphs of *stress concentration factors*, which often do not cover the exact component geometry being used. Probably the most important role for FEA in this field is to replace and supplement these graphs. Material data is then introduced in the form of tests to total separation on small smooth specimens, plotted as life  $N$  against some nominal stress  $S$ . Comparing the stress-time history at the chosen critical point with this S-N curve allows a life estimate for the component to be made.

In a variation of this method the S-N plots are obtained for whole components. Life estimates are then limited to that component, and a new test programme should strictly follow even minor changes in its geometry. This is not a very economic procedure for modern development programmes and another major role for FEA is to avoid the need for it.

A special case of component testing concerns welds in metals. Extensive tests have resulted in S-N data published in standards like BS7608, which relate total life to a nominal stress remote from the weld. This is not basic materials data, as is often implied, but component data, specific to a certain type of weld. The role of FEA is then to provide the *nominal*, or *reference*, stress-time history at a place where stresses remain elastic.

The S-N method assumes the structure to be fully elastic, not just in structural terms, but even in local fatigue-related details such as notches. It should therefore only be used in such limited cases. It is therefore only applicable to (low load-long

life) *high cycle fatigue* (HCF) problems. In application to FE models, linear elastic stresses from FE analysis can be used directly to calculate fatigue damage.

It is worth pointing out that whenever someone calculates an “allowable stress” as a percentage of yield or ultimate strength, then maybe reduces the value for the surface finish (or processing), or maybe reduces the value for cycle type (0-Max, reversed, etc.), they are actually doing a Stress-Life calculation. Some designers do this, without actually constructing a Goodman diagram. “Goodman diagrams” and more generally, mean stress effects, are covered in section 4.5.

### 3.2.2 Strain-Life (Crack-Initiation or Critical-Location Approach)

In applying the S-N approach it is assumed that stresses remain elastic in all parts of the component, even the location where a crack will start. This implies that all stresses are low, implying in turn that long lives are the aim. Certainly the method should be confined to lives greater than 10000 applications of load, and is probably better confined to lives over 100000. At shorter lives, higher loads, multiplying a nominal stress by a concentration factor will give figures greater than the yield stress at the critical location, so that yielding will occur. Techniques have therefore been developed to use the strain response in the structure for such *low cycle fatigue* (LCF) problems. Methods of predicting a strain-life history under reversed yielding were well developed before FEA became a general design tool. These use empirical relationships that are well-trying, and although non-linear FEA could replace them in some cases this is not the general practice. FEA is used instead to give better elastic predictions at points near to the critical location. These are then converted to elastic-plastic predictions at user chosen critical locations.

When a strain-time history has been determined for the critical location, or locations, material properties are introduced as data from tests conducted on small smooth specimens under different ranges of constant strain. These tests are terminated when a small crack is present, and the predicted life is regarded as the life to crack initiation  $N$ . Within some larger organizations, test specimens are sometimes cut from actual components to account for manufacturing effects when obtaining material properties. Also, some casting suppliers often cast their own specimens from production batch material.

Although the method appears to be only useful for components with short lives it has wider applications. In service a component may have quite a long life requirement but experience only rarely a load that causes yielding at the critical location. The large number of small loads it carries may well cause no damage at all, but Strain-Life methods are still then appropriate. The Strain-Life approach can also still be used to advantage even in high cycle applications, due to its less scatter-prone materials data.

The Strain-Life approach is a more generally applicable method than S-N and is used widely (especially in the automotive industry) where engineers are trying to design components to a finite life. The Stress-Life approach is found to be a subset of the Strain-Life approach since at long lives, and elastic stresses, the two methods tend to be effectively the same. Strain-Life materials data is inherently less scatter prone than S-N, so this gives the engineer an immediate advantage.

### 3.2.3 Crack Propagation Models.

If the crack propagation phase of life,  $N_p$ , is to be taken into account, crack prediction models are needed for two tasks. The first is to predict the rate of growth of a crack in, say, mm/(load application). The second is to predict how long a crack can be before the next peak in the loading history causes catastrophic propagation. Both of these are handled by Fracture Mechanics, usually Linear Elastic Fracture Mechanics. The controlling factor in both cases is the *crack tip stress intensity factor*, which depends on the crack length, the nominal stress near the crack tip and a factor  $Y$ , sometimes called the *Compliance-Function*.  $Y$  depends on component geometry and deriving expressions for it is difficult because of the singularity at the crack tip. The stress at the tip is plastic and in elastic analyses tends to infinity. Tables and graphs for it exist, as with stress intensity factor. The role of FEA in Crack-Propagation is therefore similar in some ways to its role in Stress-Life estimates since it replaces existing data banks. In fact FEA has a wider role in Crack-Propagation than in Stress-Life partly because the data banks are less well developed but particularly because  $Y$  nearly always changes as the crack grows. If this is described by a simple expression numerical integration may be good enough. In many practical cases, though, the crack extends into a region with completely different geometry, and the adaptability of FEA becomes essential.

A common procedure for crack life estimation is to assume that a crack-like feature of a certain length is present when the component is put into service. The crack initiation life,  $N_i$ , is then zero. This procedure is particularly common when dealing with welds.

### 3.3 Linking Life Estimation Methods With Design Philosophies.

Safe-Life design normally uses Stress-Life analysis, and the usual Damage-Tolerant design uses Crack-Propagation, but these associations are not rigid. Structural deterioration might be measured by a factor other than crack length, and a Safe-Life design could aim to take a component out of service well before a crack could initiate. Strain-Life is the only working method we have at present for predicting so-called life to crack initiation.

### 3.4 FE Hints and Tips – Alternative Philosophies

The task of performing fatigue calculations can sometimes appear to be very complex. However, it is worthwhile remembering that the most useful fatigue calculation is very often the simplest one. Multiaxial fatigue calculations, for instance, have a role in some specialised design situations. However, if one tried to apply such techniques to standard, and more straightforward design situations the possibility of incorrectly applying the method far outweighs any possible loss of accuracy. Remember that there is inherent scatter in fatigue results anyway and so being within a factor of 2-3 on life is usually acceptable.

Furthermore, as a routine quality assurance operation it should always be possible to undertake hand, or other types of simple calculations, to verify adopted approaches. If this is not possible then extreme caution should be exercised.

The terms durability, reliability and fatigue are often loosely used in the same analysis situation. The term *durability* is often used as a term describing the overall life requirement, such as to last for 100,000 miles. The term *reliability* usually also includes a reference to a probability of failure, such as to have a 95% likelihood of survival. The term *fatigue* is usually used to describe the process by which cracks form and grow within materials when subjected to fluctuating stresses and strains.

The design philosophy used will often depend on the availability of FE model input loading data and this will depend on what previous model data is available from the test laboratory and the availability of analytical loads models such as ADAMS or DADS. Where these are present within a company it may be possible to integrate the techniques into the overall process in such a way that fatigue calculations can be performed well before the first metal has been cut.

One final point worth remembering concerns safety factors. In the next section we will deal with material curves that are usually plotted as mean (50% failure) curves. That means that if these material curves were used in a design situation where no other safety factors were present then a subsequent 50% failure rate in service would be expected. Of course this does not happen because the factors of safety are usually introduced elsewhere. For instance, input loading curves often have safety margins built in, or sometimes, predicted fatigue lives must be 6 times larger than the require one. One important benefit of FE based fatigue design is the ability to properly manage these load factors in a methodical way. For instance data scatter, if known, can be included directly in the fatigue calculations. It is thus possible to include safety factors wherever, and whenever required, such as on loads, materials, geometry etc and to assess the sensitivity of the results to errors in any of these inputs.



## 4 The Stress-Life (S-N) Approach.

As pointed out earlier, the Stress-Life approach assumes that all stresses in the component, even local ones, stay below the elastic limit at all times. It is the oldest of the three main methods (19<sup>th</sup> century) and is still suitable when the applied stress is nominally within the elastic range of the material and the number of cycles to failure is large. The nominal stress approach is therefore best suited to problems that fall into the category known as *high-cycle fatigue* (HCF). The nominal stress method does not work well in the *low-cycle fatigue* (LCF) region where the applied strains have a significant plastic component. In this region, a strain-based methodology must be used.

### 4.1 Defining Stress Cycles

Figure 19 gives the basic parameters that define this type of loading

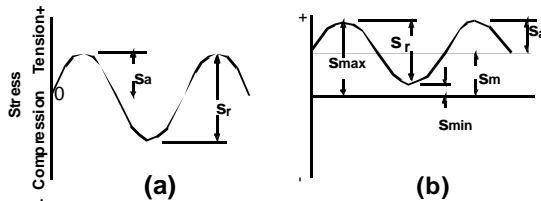


Figure 19. Examples of some stress cycles, (a) fully reversed, and (b) offset

Figure 19(a) shows a fully reversed stress cycle with a sinusoidal form. This is an idealised loading condition typical of that found in rotating shafts operating at constant speed and constant load. The maximum and minimum stresses are of equal magnitude but opposite sign, tensile stress being considered positive and compressive stress negative. Figure 19(b) illustrates the more general situation where the maximum and minimum stresses are not equal in magnitude. In this case they are both tensile and so define an offset for the cyclic loading. It is therefore convenient to define a fluctuating stress cycle by two components, a static or mean state stress  $S_m$ , and an alternating stress amplitude,  $S_a$ . It is sometimes necessary to consider the stress range, which is the algebraic difference between the maximum stress in a cycle,  $S_{max}$ , and the minimum stress,  $S_{min}$ . Stress amplitude is one half the stress range and mean stress is the algebraic mean of  $S_{max}$  and  $S_{min}$ . The same



information can be conveyed by specifying either  $S_{max}$  or  $S_{min}$  and the stress ratio  $R$ , which is  $S_{min} / S_{max}$ .

## 4.2 The S-N Curve

Users of FEA will be familiar with the fact that holes, grooves, fillets and other geometrical features cause high local stresses under load, called stress concentrations. It follows that any attempt to measure the basic fatigue properties of a material, achievable under ideal conditions, must use specimens free of these features. At one time the most common test was a cylinder with very slow changes of section, loaded in bending and with a polished surface in the region where cracks were likely to start. This rotating bend or Wöhler test has limitations and we now prefer a cylinder loaded in axial tension, again free of sudden changes of geometry and with a polished surface at the critical section. In either case a number of identical specimens are tested to total separation and the number of cycles needed is recorded as  $N$ . Load, not stress, is kept constant during the test. For each specimen a nominal stress,  $S$ , is calculated from simple elastic formulae and the results are plotted as the **un-notched S-N diagram**, a basic material property.

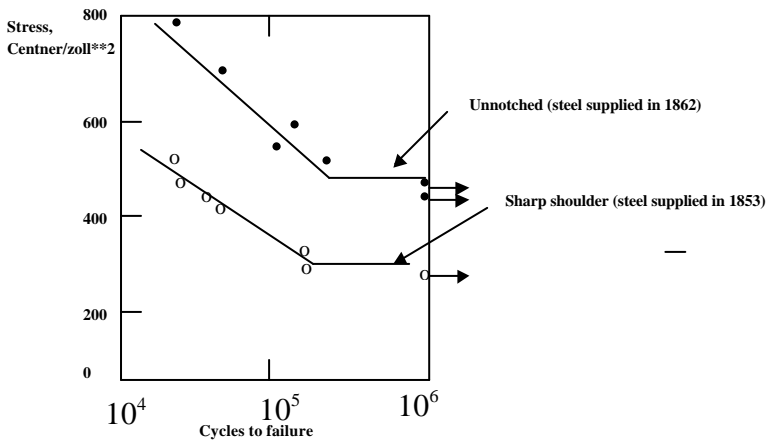


Figure 20. S-N data reported by Wöhler.

(Note, 1 centner = 50 Kg, 1 zoll = 1 inch, 1 centner / zoll<sup>2</sup> » 0.75 MPa.)

$N$  is always plotted on the x-axis, and a logarithmic scale is used. The y or  $S$  axis may be linear or logarithmic, but logarithmic is becoming the norm. The mean line in the finite-life region (10000 to 10 million cycles) is then usually straight (Figure 21), giving the convenient relationship :-

$$N = aS^{-b}$$

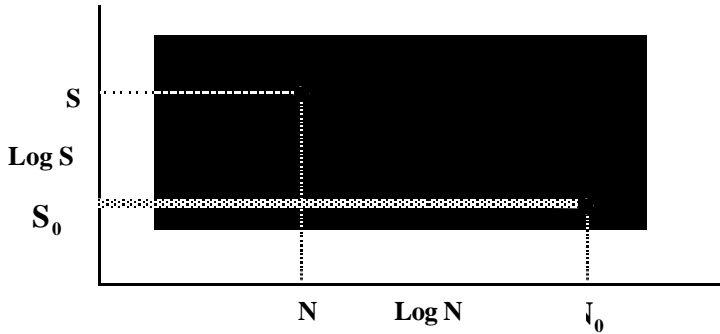


Figure 21. Standard form of the material  $S$ - $N$  curve

The inverse slope of the line is  $b$ , called the Basquin exponent, and  $a$  is related to the intercept on the  $y$  axis. Given any two points on the line  $a$  and  $b$  are easily calculated. Other symbols are sometimes used.

The value of  $b$ , for a particular  $S$ - $N$  slope, gives a good indication of how accurate the estimate of stress at a critical location needs to be to give a reliable life estimate. If  $b$  is 10 then a 7% change in stress causes a 100% change in life. This is why using FEA for fatigue calculations is more demanding than using it in a static design.

Some metals, particularly low alloy steels, have a two-line  $S$ - $N$  plot, with flattening of the relationship when  $N$  is greater than about 10 million cycles. The line may become horizontal so that no failures occur at higher values of  $N$  and the material is said to have a **fatigue limit**,  $S_0$ , which is important if infinite life is the aim. However, great care should be taken because this can be sensitive to a variety of effects such as mean stresses and corrosion. The treatment of  $S_0$  has had a lot of attention, and the convention for any particular class of components will be dealt with in the software, but it is desirable for the user to know the convention. Welds in ferrous metals, for instance, are assumed not to have a fatigue limit but experience a change of slope at  $N=10^7$

For materials which do not exhibit a true fatigue limit, tests are usually terminated at between  $10^7$  and  $10^8$  cycles. The corresponding  $S$  is then quoted as an **endurance limit** at the specified  $N$ . Service failure may then occur if more than  $10^8$  cycles are applied. Structural alloys of aluminium are well known as materials prone to this effect. There is no strict convention about the use of the terms fatigue limit and endurance limit, and the original test documentation should be examined if the difference is critical.

### 4.3 Limits of the S-N Curve

The S-N approach is applicable to situations where all stresses, even local ones, remain elastic. In practice this means that the S-N curve should be confined on the life axis to numbers greater than about 10,000 cycles. Figure 22 shows typical S-N curves for both ferrous and non-ferrous metals. It is important to note the limits of the log  $N$  axis, the presence of a fatigue limit for the mild steel and the absence of a fatigue limit for the aluminium alloy. Because both materials represented have relatively low yield stresses, the life axis is confined to begin at  $10^5$  cycles at which point the alternating stress is about 350 and 300 MPa respectively for the two alloys.

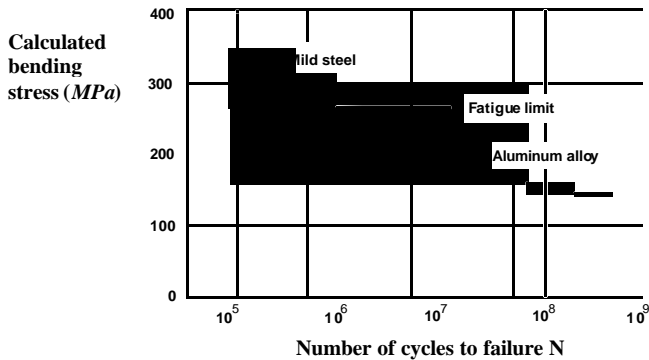


Figure 22. Idealised S-N curves for ferrous and non-ferrous metals

### 4.4 The Role of Stress Concentration.

When determining the basic S-N properties of a material, care is taken that the specimens used are free from geometrical factors which would cause high stress gradients and so create local regions of high stress. Real components must have holes, grooves, changes of section etc. which will cause local “hot spots” of high stress. Fatigue will start at these hot spots and life calculations must allow for their effect. It is in making this allowance that FEA methods differ most from traditional ones. In both cases some way must be found of converting applied loading into local stresses at the point where a crack is likely to start. In the traditional approach features causing high local stresses are called “notches” and a key factor in dealing with them is the **stress concentration factor**,  $K_T$ . For any given geometry, like a circular hole for instance, this is defined as:

$$K_T = \frac{\text{Maximum stress in the region of the notch}}{\text{Nom stress remote from the notch}}$$

Closed-form solutions exist for many of the simple common notches, and extensive tables and graphs have been published for the more difficult ones that are of practical importance. The fatigue analyst's task is then to identify a likely site of failure, identify a suitable place remote from that site where nominal stress can be calculated, and find an expression for  $K_T$  which allows the two to be related. With FEA it is possible to go directly from applied loads and component geometry to stress histories at likely sites of failure. This completely alters the basis of calculation, although it may still be convenient to retain some of the traditional approach. FEA, for instance, can be used to calculate  $K_T$  values. This is useful for engineers who find them more familiar, or can visualise the effect of geometry changes better in this way. The versatility of FEA would still provide an advantage unless the notch is a simple one. Combinations of notches close together would usually favour FEA, for instance. In some circumstances it may still be advantageous to use FEA to find a nominal stress remote from the notch and then use a well-tried  $K_T$  figure to convert this to a local stress. It is not always practicable to provide detailed meshes for local features in large models such as of a ship or an aircraft.

The concept of  $K_T$  gives a useful way of visualising some of the basics of fatigue life estimation. Consider a cylindrical specimen loaded in bending. The specimen may have a cylindrical groove with a  $K_T$  of 2.25. Specimens without the groove could be considered un-notched, so the effect of introducing the groove is to multiply the stresses at the failure location (the bottom of the notch) by 2.25. If we test specimens with and without the groove we would therefore expect the grooved (notched) tests to lie on a line which is the un-notched line with all stresses divided by 2.25. Figure 23 compares test data with this line, and shows a divergence which usually occurs. The test results have longer  $N$  values than predicted, and the full damaging potential of the groove is not being realised. One of the objectives of modern life estimation methods is to remove some of the conservatism implied when  $K_T$  is used like this.

One simple but incomplete way forward is to introduce a parameter called the **strength reduction factor**. For any given life  $N_f$  this is defined as:-

$$K_f = \frac{\text{Stress needed to cause failure in an un-notched specimen}}{\text{Stress needed to cause failure in a notched specimen}}$$

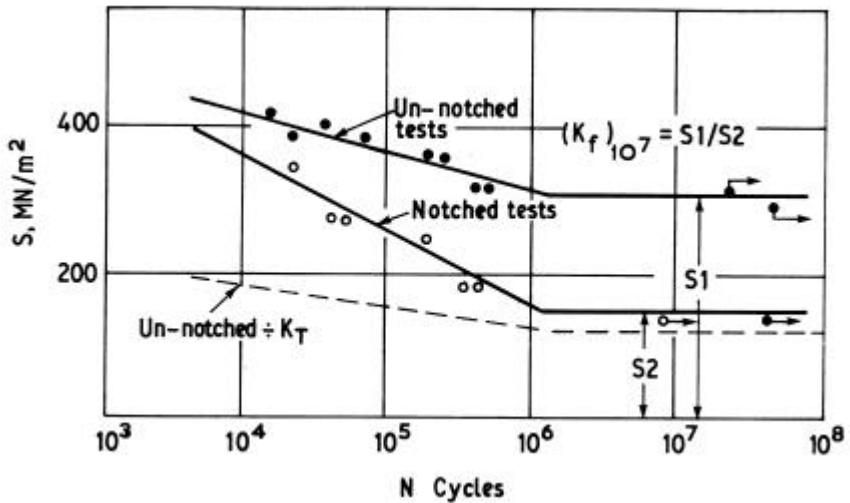


Figure 23. The role of  $K_T$  and  $K_f$

Examination of Figure 23 shows that unless the notched and un-notched lines are parallel in the region below about 10 million cycles  $K_f$  will depend on the choice of  $N_f$ . In addition  $K_f$  can only be calculated if tests on the particular notch being studied are available, and each type of notch will need a new set of tests. The popularity of  $K_f$  as a parameter dates from a period when infinite life was the objective and the fatigue limit was the most important parameter being measured. It is of doubtful use in modern circumstances.

Empirical expressions have been published for the relationship between  $K_T$  and  $K_f$ , often given in publications like Peterson, which are primarily lists of  $K_T$  values. A *sensitivity index*  $Q$  showing how sensitive a material is to the presence of notches is also quoted at times. These approaches are of restricted use, although it is worth noting that metals with low ductility will have  $K_f$  close to  $K_T$ . Attempts to express  $Q$  as a quantity will depend on the position on the  $N$  axis chosen for the calculation, and are bound to be limited. Note that in dealing with  $K_T$  and  $K_f$  FEA can only contribute directly to  $K_T$ . Any  $K_f$  values derived will depend on the reliability of the empirical relationships used.

#### 4.4.1 Example: Simple Fatigue Strength Estimates

Un-notched specimens of a particular metal have a mean  $S-N$  line which follows the relationship  $N = aS^{-b}$ . Find  $a$  and  $b$  if  $N = 10^8$  when  $S = 300 \text{ MNm}^{-2}$  and  $10^6$  when  $S = 600 \text{ MNm}^{-2}$

Taking logarithms:

$$8 = \text{Log } a - b \times \text{Log}(300)$$

$$6 = \text{Log } a - b \times \text{Log}(600)$$

$$\text{giving } b = 6.64; a = 2.87 \times 10^{24}$$

The material is used in a component with a groove having a  $K_T$  of 2.25. Estimate the allowable  $S$  for  $N = 10^7$  cycles if the full effect of the notch is realised. What will be the value of  $S$  if the actual strength reduction factor is only 2.0?

Allowable stress for  $10^7$  life on un-notched specimens

$$S^{6.64} = 2.87 \times 10^{17} \quad \text{giving } S = 425 \text{ MNm}^{-2}$$

(a) Full  $K_T$  nominal amplitude =  $(425/2.25) \text{ MNm}^{-2} = 189 \text{ MNm}^{-2}$

(b) Only  $K_T$  nominal amplitude =  $(425/2) \text{ MNm}^{-2} = 212.5 \text{ MNm}^{-2}$

#### 4.5 The Influence of Mean Stress

Fatigue life depends mainly on the amplitude of stress or strain existing in the component, but this is modified by the mean value of stress. Many components carry some form of “dead load” before the working stresses are applied, and some way of allowing for this is then needed. Traditional methods use stress as the controlling factor. The general trend is quite simple. For a given life the allowable amplitude of fatigue stress gets smaller as the mean stress becomes more tensile, and to a lesser extent increases when the mean stress is compressive. The latter effect is of great practical importance because processes like cold-rolling, shot-peening, etc, are used to deliberately introduce compressive mean stress in surfaces and so improve fatigue resistance. When it comes to quantitative predictions it must be admitted that we only have any substantial experience in the tensile region, and the problem reduces to choosing the best of the available formulae. Often only tests at one mean stress are available, and the mean value is usually zero.

One simple way forward is to assume that mean stress reduces allowable applied amplitude of stress in a linear way. It is reasonable to expect that once the mean stress reaches the ultimate tensile strength of the material no fatigue load can be carried at all. If we know the fatigue strength at any mean we can then define the line. This is known as Goodman's Rule. Above a certain mean stress combining the mean stress with the fatigue loading takes the material beyond its yield stress at every stress peak. To allow for this a cut-off line can be introduced, giving the Modified Goodman Rule. With hindsight it is easy to see that this approach could

not properly allow for material behaviour at notch roots, but as an empirical formula the Goodman approach has been relatively successful.

There are other rules which have been found to fit experimental data better. A possible general formula for the un-notched case is (see Figure 24):

$$S_a = S_0 \left[ 1 - \left( \frac{S_m}{S_u} \right)^{m1} \right]$$

where  $S_a$  is the allowable stress amplitude for a given life when the mean stress is  $S_m$ . Other factors in the equation are:

$S_u$  = Ultimate tensile strength

$S_0$  = Allowable stress amplitude at zero mean stress (for stated N, usually  $10^7$ )

$m1$  = An arbitrary constant.

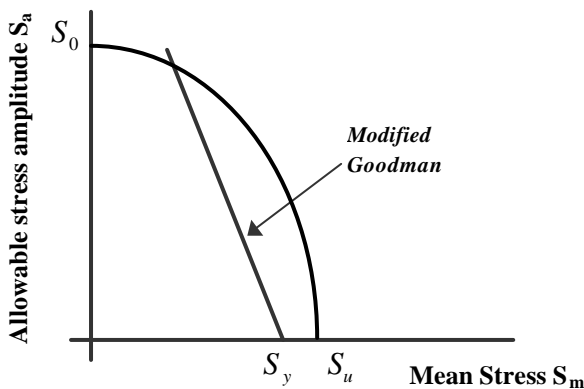


Figure 24. The effect of mean stress

The form of the relationship then changes with  $m1$ . If tests at more than one mean stress are available  $m1$  can be calculated. Otherwise the most conservative assumption is to put  $m1 = 1$ , reverting to the Goodman formula, and there is experimental support for not exceeding a value of 2 (the Gerber parabola). Some references such as the Engineering Science Data Units (ESDU) sheets suggest a figure of approximately 1.5 for most steels.

Whatever expression is used will predict the reduction in alternating stress needed to give a chosen life when a certain mean stress is present. The most common life is  $10^7$  cycles or greater. If tests at shorter lives are available lines can be derived for the change in alternating stress at this life. This is known as a ***Haig diagram***, and a plot showing a number of lines, each for a different value of N, is called a Haig Master Diagram.

#### 4.5.1 Example: Correcting For Mean Stress Effects

A component undergoes an operating cyclic stress with a maximum value of 759 MPa and a minimum value of 69 MPa. The component is made from a steel with an ultimate strength  $S_u$  of 1035 MPa, an endurance limit  $S_e$  (at  $10^6$ ) of 414 MPa and a fully reversed stress at 1000 cycles,  $S_{1000}$  of 759 MPa.

Plot a Goodman diagram with 2 constant life lines on it corresponding to  $10^3$  and  $10^6$  cycles. These must both go through  $S_u$  on the zero mean stress amplitude (x) axis and the appropriate points on the stress amplitude (y) stress axis which are the endurance limit,  $S_e$ , and  $S_{1000}$  values (see Figure 25 below).

A 3rd line can be drawn through  $S_u$  (on the x axis) and another point defined by the operating stress.

$$S_a = \frac{S_{\max} - S_{\min}}{2} = \frac{759 - 69}{2} = 345 \text{ MPa}$$

$$S_m = \frac{S_{\max} + S_{\min}}{2} = \frac{759 + 69}{2} = 414 \text{ MPa}$$

When the stress conditions for the component ( $S_a = 345$  MPa,  $S_m = 414$  MPa) are plotted on the Goodman diagram, the point falls between the  $10^3$  and  $10^6$  life lines. This indicates that the component will have a finite life, but the life is greater than 1000 cycles. This 3rd line intersects the fully reversed alternating stress axis at a value of 573 MPa.

By taking one vertical (zero mean stress) slice through this diagram the equivalent (zero mean stress) S-N diagram can be envisaged (see Figure 26).

The value for  $S_n$  can now be entered on the S-N diagram to determine the life of the component  $N_f$ . (Recall that the S-N diagram represents fully reversed loading). When a value of 573 MPa is entered on the S-N diagram for the material used for



the component, the resulting life to failure can be obtained graphically as  $N_f = 2.4 \times 10^4$  cycles.

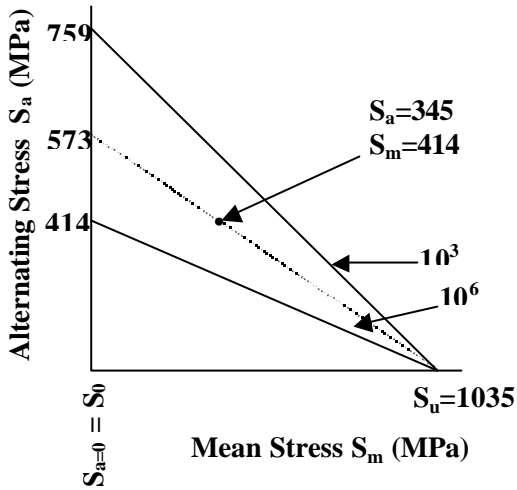


Figure 25 . Goodman diagram

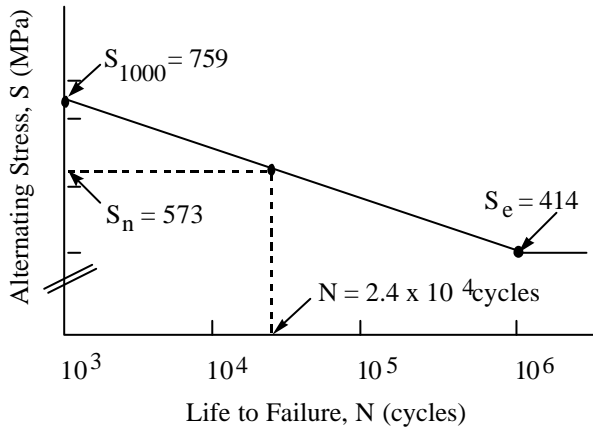


Figure 26. S-N diagram generated from Goodman diagram

Alternatively, it will be recalled that the S-N curve is given by equation,

$$N = N_0 \left( \frac{S}{S_0} \right)^{\frac{1}{k}}$$

where  $k = -(1/b)$ . So, for the conditions defined at  $10^3$  and  $10^6$  cycles,

$$k = \frac{-(\log S - \log S_0)}{(\log N_0 - \log N)}$$

Therefore 
$$k = -\frac{1}{3} \log\left[\frac{S_3}{S_6}\right] = -\frac{1}{3} \log\left[\frac{759}{414}\right] = -0.088$$

and so for  $S_n = 573$ , the life can be calculated from  $N = 10^6 \left(\frac{573}{414}\right)^{-11.4}$

Therefore, N is approximately  $= 2.4 \cdot 10^4$ .

#### 4.6 Variable Amplitude Response – Block Loading and Palmgren-Miner

The analysis so far has assumed that the fatigue loading is constant-amplitude with or without a mean offset. However, it is more common for the responses to vary in magnitude. The simplest extension of the constant amplitude case is one in which the amplitude of the sine waves changes from time to time, as in Figure 27.

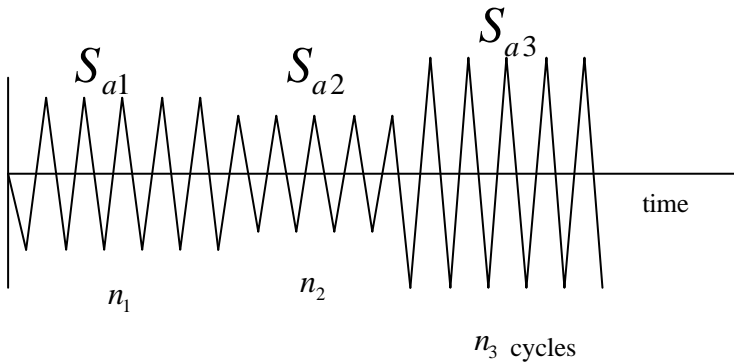


Figure 27. A typical block loading sequence

The history then consists of  $n_1$  cycles of amplitude  $S_1$ ,  $n_2$  of  $S_2$ ,  $n_3$  of  $S_3$  and so on. Usually the pattern repeats after a small number of  $S$  values, say  $S_n$ . The sequence up to  $S_n$  is then called a “block”, and the target is to estimate how many of these “blocks” can be applied before failure occurs. The rule generally used is the **Miner or Palmgren-Miner** hypothesis. Considering first the  $n_1$  cycles of  $S_1$ , if we have S-N data we can find the number of cycles of  $S_1$  which would cause failure if no other stresses were present. Calling this  $N_1$ , the simplest assumption is then that  $n_1$  cycles of  $S_1$  use up a fraction  $n_1/N_1$  of the life. Doing a similar calculation for all

the other stresses and summing all the results gives the total damage fraction for one block. The reciprocal of this is then the life in blocks. Given as an equation this is:-

$$\sum \frac{n}{N} = 1.0$$

Or, for the sequence in Figure 27 this would be  $\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \dots = 1.0$

The limitations of Miner's Hypothesis are that it is:-

(i) Linear, i.e. it assumes that all cycles of a given magnitude do the same amount of damage, whether they occur early or late in the life.

(ii) Non-interactive (sometimes referred to as sequence effects) i.e. it assumes that the presence of  $S_2$  etc. does not affect the damage caused by  $S_1$

(iii) Stress-independent i.e. it assumes that the rule governing the damage caused by  $S_1$  is the same as that governing the damage caused by  $S_2$ . This limitation is often misunderstood, and is sometimes confused with (ii).

These assumptions are known to be faulty, but wide use of the hypothesis has shown that in most circumstances it gives acceptable results. A device sometimes used to make the comparison a more general one is to use  $E[D]$  for expected damage, so that:-

$$\sum \frac{n}{N} = E[D]$$

For conservative design  $E[D]$  is put  $< 1.0$ .

#### 4.6.1 Example: Palmgren-Miner Cumulative Damage Calculation

Part of a steel structure, located permanently in the sea, is known to be susceptible to fatigue damage. Strain gauges attached to this part were monitored continuously during the first year of service producing the following informaton.

Strain range recorded	Derived stress range (MPa)	Number of occurrences
$1.45 \times 10^{-3}$	300	500
$1.21 \times 10^{-3}$	250	2500
$0.98 \times 10^{-3}$	203	15000
$0.76 \times 10^{-3}$	157	120300
$0.68 \times 10^{-3}$	140	400000
$0.60 \times 10^{-3}$	124	1000000
$0.54 \times 10^{-3}$	112	3000000
$0.45 \times 10^{-3}$	93	5000000

Specimen tests on the same material showed that the fatigue limit in air was 156 MPa and that in seawater it was  $110 \text{ N/mm}^2$ . In addition it was found that stresses above either of these levels produced failure according to the following relationship.

$$N = 2.196 \times 10^{25} \cdot S^{-8.3333}$$

It was originally intended that the structure should be protected from corrosion through its entire life. Determine the life under these conditions.

During first seven years, damage occurs only at a strain range of  $0.76 \times 10^{-3}$  (157MPa – smaller cycles being below the fatigue limit) and above. Total damage during this period =  $0.0437 \times 7 = 0.3059$ . Therefore, remaining damage to be accumulated =  $1 - 0.3059 = 0.6941$ .

$\Delta e$	$\Delta S$	$n$	$N$	<i>n/N without corrosion</i>	<i>n/N with corrosion</i>
$1.45 \times 10^{-3}$	300	500	49,700	0.01	0.01
1.21	250	2,500	225,000	0.0111	0.0111
0.98	203	15,000	$103 \times 10^6$	0.0115	0.0115
0.76	157	120,300	$10.8 \times 10^6$	0.0111	0.0111
0.68	140	400,000	$27.4 \times 10^6$	0.0	0.0146
0.60	124	1,000,000	$77.7 \times 10^6$	0.0	0.0129
0.54	112	3,000,000	$187 \times 10^6$	0.0	0.0160
0.45	93	5,000,000	$854 \times 10^6$	0.0	0.0

If corrosion fatigue occurs, amount of damage/annum = 0.0872. Therefore, remaining life after breakdown of protection is

$$\frac{0.6941}{0.0872} = 7.96 \text{ years}$$

So total life = 7 years + 7.96 years.

Original intended life = 22.9 years (1/0.0437) without corrosion. The above example demonstrates the reduction due to subsequent corrosion.

#### 4.7 Variable Amplitude Loading – Rainflow Cycle Counting

Many real engineering components experience stress responses that are more complex than this. Consider the sequence shown in Figure 28.

This is a commonly occurring situation where dual mode response is present in a structure. Each cycle transition is equal to approximately 100MPa but the overall transition of the peaks is over 400MPa. The difficult question here is what ‘cycles’ of stress to use? One approach would be to take the stress difference between adjacent peaks and troughs. This would result in many cycles of about 100MPa. However, because fatigue behaviour is non-linear, higher stress levels cause much higher fatigue damage. The above approach would therefore grossly underestimate fatigue damage.

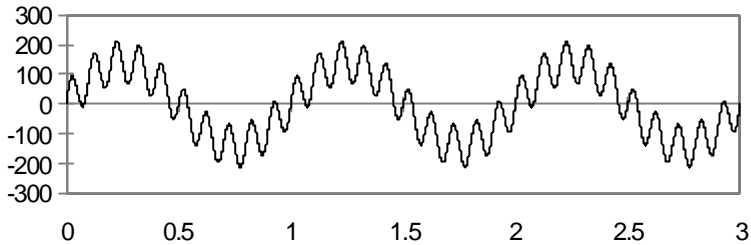


Figure 28. Counting Rainflow cycles from an irregular time history

An alternative approach is to assume the peak levels are representative of the stress amplitudes. If we used this amplitude for all cycles, then this approach would grossly overestimate damage.

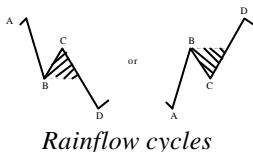
Instead, a technique is required which can identify overall trends in the response, whilst also keeping track of intermediate and small response cycles properly. **Rainflow ranges** have been widely used for estimating fatigue damage from random signals since Matsuishi and Endo first introduced the concept to the scientific community over twenty years ago.

The procedure for Rainflow cycle counting is relatively straightforward. Software will incorporate a tested algorithm for counting Rainflow cycles, but users should understand the logic and confirm their understanding by solving a short sequence. The most common procedure is:-

- [1]. Extract peaks and troughs from the time signal so that all points between adjacent peaks and troughs are discarded.
- [2]. Make the beginning, and end, of the sequence have the same level. This can be done in a number of ways but the simplest is to add an additional point at the end of the signal to match the beginning.
- [3]. Find the highest peak and reorder the signal so that this becomes the beginning and the end. The beginning and end of the original signal have to be joined together.

[4]. Start at the beginning of the sequence and pick consecutive sets of 4 peaks and troughs. Apply a rule that states,

If the second segment is shorter (vertically) than the first, and the third is longer than the second, the middle segment can be extracted and recorded as a Rainflow cycle. In this case, B and C are completely enclosed by A and D.



[5]. If no cycle is counted then a check is made on the next set of 4 peaks, ie peaks 2 to 5, and so on until a Rainflow cycle is counted. Every time a Rainflow cycle is counted the procedure is started from the beginning of the sequence again.

Eventually all segments will be counted as cycles and so for every peak in the original sequence there should be a corresponding Rainflow cycle counted. There will be 5 cycles obtained from the 10 peaks and troughs.

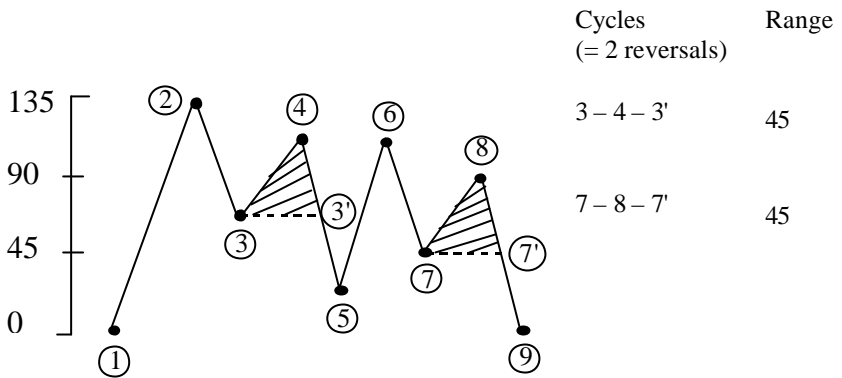
#### 4.7.1 Presentation of Rainflow Counts

Sometimes only the range of a Rainflow cycle is recorded. In this case a graph or two-dimensional histogram describes the distribution. If the mean of each cycle as well as its range is recorded a three-dimensional histogram like Figure 89 is a common way of presenting the data.

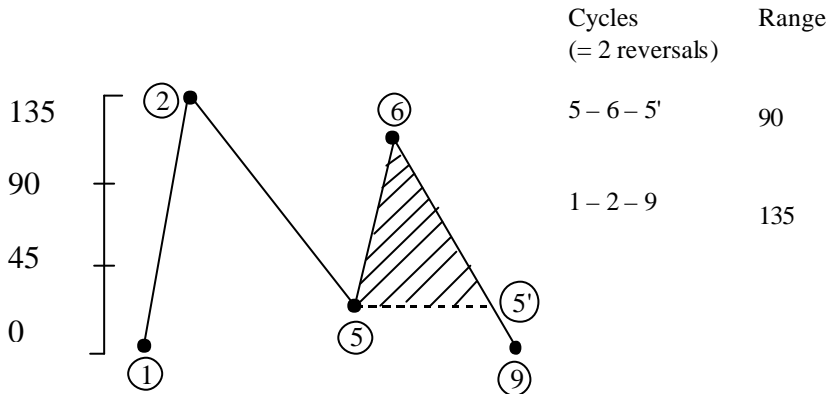
#### 4.7.2 Example: Rainflow Cycle Counting

As an example of this let us extract Rainflow cycles from the following sequence.

Peak	1	2	3	4	5	6	7	8	9
Value	0	135	67.5	112.5	22.5	112.5	45	90	0



Then reduces to:-



This gives a total Rainflow count of:-

Range	45	90	135
Reversals	4	2	2

#### 4.8 Material and Component S-N Curves

In component S-N analysis, including some welding standards, locations of **reference** stresses are often defined at a fixed distance from a geometrical feature. Sometimes they are referred to as **nominal** stresses but more generally they are reference stresses. In this case, it is only the stress at this location that is required to

quantify a fatigue life for the component. In spot weld analysis, forces and moments in BAR elements can be used to calculate stresses for each spot weld based on the thickness of the sheets and the size of the weld nugget. The actual local stresses in the model are not used, only these derived nominal stresses.

#### **4.8.1 Material S-N Curves**

Material S-N curves are produced using specimens, such as hour glass specimens, where a uni-axial stress can be calculated from elastic theory and compared directly with a failure life. The specimen is free from local stress raisers and so only the nominal stress level is important. As long as similar stress conditions occur again in another specimen or component, a similar failure life would be expected.

A material S-N curve therefore relates elastic stress,  $S$ , to the number of cycles,  $N$ , required to cause failure. Such curves can be used for detecting failure locations and estimating lives across an entire finite element model for which appropriate elastic stresses have been calculated. If all other factors are the same, and there is only one load case, the failure location(s) will correspond to regions of the model exhibiting the highest stresses. Furthermore, the distribution of expected lives can then be usefully represented by a contour plot of life.

#### **4.8.2 Component S-N Curves**

Component S-N curves are generated by testing complete components, or pseudo-components, rather than smooth polished bars of material. These curves can be used to estimate how long the component, as a whole, will last under cyclic loading. The failure location is pre-defined by the component itself during the cyclic testing process. The component S-N approach is very useful in situations where an accurate description of local stress, either elastic or elastic-plastic, is difficult to achieve such as in the case of welded constructions or composite materials.

With component S-N curves the stress parameter used can be any nominal value conveniently measured during the fatigue test. The failure location is usually remote from the measurement position. Consider a through thickness fillet weld joining two plates of varying thickness as shown in Figure 29.



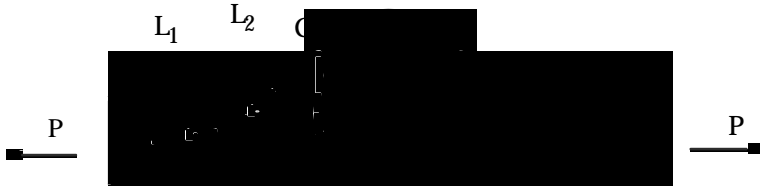


Figure 29. An example of a component used to generate a component S-N curve (fillet weld between 2 plates of varying thickness)

Now let us assume that as a result of the cyclic load,  $P$ , the component fails at position  $C$ . Let us further assume that the values of two stresses,  $S_1$  and  $S_2$ , at locations  $L_1$  and  $L_2$  respectively, are either measured, or known. It is now possible to generate 2 location specific S-N curves, similar to those given in Figure 30.

Of course each S-N curve would be different and so it is apparent that the S-N curve is now a function of the location at which the reference stress was defined, i.e. position  $L_1$  or  $L_2$ . Furthermore, different stresses,  $S_1$  and  $S_2$ , result in the same life,  $N$ , for the component with failure occurring at position  $C$ .

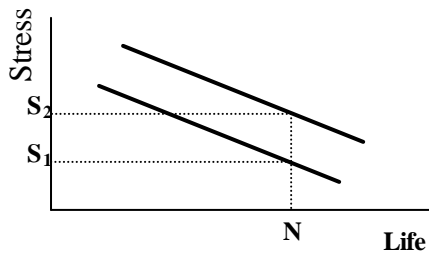


Figure 30. Component S-N curves defined for point  $L_1$  and  $L_2$

It is worth making the hypothetical comparison between

[1]. The fatigue life obtained by combining the reference stress at  $L_1$  (or  $L_2$ ) with the component S-N curve (based on  $L_1$  or  $L_2$ ).

[2]. The fatigue life obtained by combining the true stress at  $C$  with the material S-N curve for the heat affected zone (HAZ) around the weld toe.

In an ideal world the 2 answers would be the same. The main problem is how to obtain the material curve for the heat affected zone.

## **4.9 FE Hints and Tips – S-N approach**

### **4.9.1 Component S-N Curves**

If FEA based fatigue techniques are used with component S-N curves extreme care needs to be taken in determining which stress locations to use. For example, if a weld analysis is being undertaken then an FEA group needs to be defined where all the elements of the group correspond correctly with the measurement location in the component S-N test. For instance, the FEA group may be all nodes 10mm away from the weld toe. Then, the fatigue life results are only relevant to the fatigue critical location, ie, at the weld toe. In this situation, general contour plots of life, or damage, are meaningless.

### **4.9.2 Nodal vs Element Averaging**

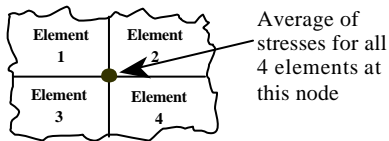
One very important question concerns the choice of stress value used and whether stress averaging over adjacent nodes or elements should be undertaken. In practice, fatigue damage is caused by peak stresses and so great care should be taken to avoid clipping these peaks with some kind of averaging process. The basic choices are (i) internal element-Gauss values, such as those shown in Figure 31(b), (ii) un-averaged nodal (sometimes called element nodal) values where the nodal values from each element are used directly and (iii) averaged nodal where adjacent values at nodal positions are used to form one average result as shown in Figure 31(a).

In general, because of the need to retain peak stress values, un-averaged nodal results should be used instead of averaged nodal, or element-Gauss values. Element-Gauss values are usually specified away from peak stress values. Consider, for example the situation where a 3D element such as the one shown in Figure 31(b) forms a surface element at a peak stress location. In nearly all practical situations the peak stress will be a maximum at the surface and so the element-Gauss values specified will always provide lower stress values. Likewise, nodes (even those at peak stress locations) will never show the peak stress if they are averaged. For this reason, element-Gauss values should generally be avoided and instead un-averaged nodal results should be used.

Some FE codes, however, do not easily allow un-averaged nodal values to be used globally and so a reasonable compromise is to use averaged nodal. In this case, several points of caution should be noted. Firstly, averaging between elements at nodal points should give reasonable results, with little clipping of peak values, as long as the FE mesh is behaving properly. Secondly, any sort of averaging where there is a change of material or element type should be avoided. Finally, it is vital that consistent coordinate systems are adopted prior to any averaging. As long as

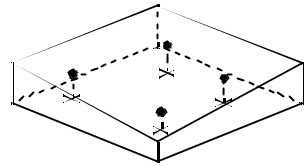
these points are followed, averaged nodal results should generally give the reasonable results.

This issue is also discussed in section 9.7.2.



Nodal values for stress can either be 'as is' or averaged between adjacent elements.

(a)



Element stresses are normally at internal positions in the element and so usually underestimate peak values.

(b)

Figure 31. Nodal vs element averaging

### 4.9.3 Spot Welds

A special case of the component S-N curve is the spot-weld S-N curve. In this case, the stress referred to is the stress calculated in the spot weld nugget, or in the upper or lower sheet metal. Sometimes the moments and forces in the spot weld are used to determine stresses. In this case coarse FE meshes (used to transfer loads only) can be used.

As always, an accurate stress prediction is important. For a component S-N curve this means that a reasonably well-refined mesh will be needed at the point of the reference stress. However, away from that point a mesh density sufficient to carry loads accurately is all that is required.

Sometimes one needs to separate bending stress from axial stress acting on the weld. Summing and differencing the stress values on the top and bottom surfaces of the shell elements can do this. It should, of course, be remembered that shell elements give both top and bottom surface stresses.

### 4.9.4 Welds

In a typical welded structure, the stress fields can be quite complex. Whilst the weld may have been characterised (and in a standard such as BS7608 formally defined) by a simple applied direct tensile stress, it may not be that simple when used in a complex model. The component, in the area of a given weld, may be subject to bending stresses, or bi-axial stresses. It is therefore important to study the stress conditions to ensure that they comply with the standard. In a good graphical post-processor it should be possible to plot the surface principal stresses as a vector (as opposed to the more popular colour contours), to ensure that their

principal direction and the bi-axiality (ratio of minimum to maximum stress) are consistent with the original assumptions. As the loading applied to the structure (in fatigue terms) may not be a simple cycle, and if there is more than 1 load applied, the stress directions (and other parameters) may vary with time. Although the highest stresses will dominate, modest stress levels must not be ignored.

Welds have an additional problem. They inherently represent a geometric discontinuity, which poses problems for the FE analysis. A discontinuity will give rise to an infinite stress, but a normal mesh cannot, and need not, resolve this. Refinement will give ever-higher stresses, to levels way beyond the material's yield and ultimate strengths. Clearly this is invalid. In practice this sharp corner is usually filled by the weld-fillet and will give rise to local yielding. It is therefore unwise to even consider the stresses in this area. It has been suggested that the under-resolved stresses at a sharp corner could be representative of a true value, but this is most unreliable. The approach of using a Component S-N curve avoids these problems. Welds also represent a metallurgical discontinuity, but this is handled by the component S-N curve.

Some standards do not use a single reference point. They use 2 reference points at set distances from the weld and then derive a stress based on some assumed gradient from the first point, through the second, towards the weld. This becomes the reference value.

At times a weld may be quite long and subject to different stresses at each end. In this case the fatigue life may be different at each end (and intermediate points). Then, instead of a single reference point, the weld should be considered as a series of sections each with its own reference point, building into a reference strip, running parallel to the weld. The principles are otherwise the same.

When using a standard it is important to refer to that standard for full details and limitations.

One particular method, RIMS for seam welds, has been set out by Koettgen et al 1992. To use this approach, a constant notch radius  $R$  is modelled at the intersection of the fusion zone and base material and this defines the nominal weld geometry. One set of material properties is used for the welds using a local Stress-Life approach, back calculated from tests, and including the effects of mean loads and scatter. A detailed finite element model is used to compute the surface stress at the critical locations at the joint, in this case using a 1 mm notch root radius (the 1mm notch radius gives the "1" in RIMS.) Based on a nominal section, a stress concentration factor can be defined  $K_{T,r=1mm}$ . From fatigue tests of the components the endurance limit range can be determined, ( $?S_E$ ). The weld material endurance limit range (a pseudo stress,  $?s_E$ ), is then back calculated. This procedure is shown in *Figure 32*.

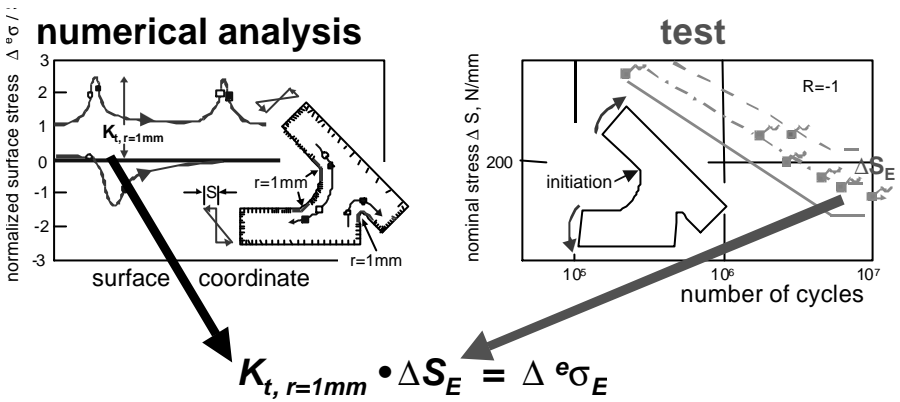


Figure 32. The RIMS approach as set out by Koettgen et al, 1992

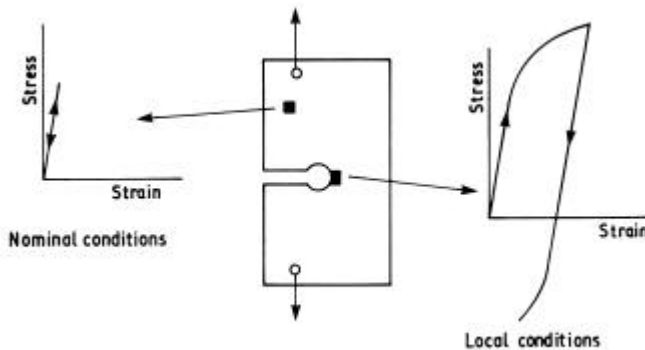
#### 4.9.5 $K_T$ and $K_f$ Values Within an FE Model

If stress concentrations within the model are not adequately detailed then it may be necessary to apply appropriate  $K_f$  (not  $K_T$ ) values to these regions in order to obtain realistic fatigue lives. On the other hand, if a fatigue critical location, such as a notch, is modelled perfectly within an FE model it is likely that the stress value obtained is a  $K_T$  one and not a  $K_f$  one. It may be necessary, therefore, to consider applying factors less than 1.0 to such regions in the model to recreate more realistic stress values in line with the  $K_f$  functions described in section 4.4. This is only a minor point since the difference between  $K_T$  and  $K_f$  is rarely considered in practice.

## 5 The Strain-Life (e-N) Approach.

An obvious way of improving the correspondence between a fatigue life estimate and the physical processes that are taking place is to split the life into crack initiation and crack propagation. In modern software the dominant method for the so-called crack initiation phase is the Strain-Life approach, sometimes called the Critical-Location Approach (CLA), Local-Stress-Strain, or Crack-Initiation. However, as was pointed out earlier, the definition of “crack initiation” used by proponents of the Strain-Life method is often in conflict with that of metallurgists who believe that most materials have pre-existing flaws (cracks) in them making the concept of crack initiation dubious. The Strain-Life method concentrates on the strain-time history at the point where failure is likely to occur. This conforms to modern observations about the mechanism of fatigue damage that show that fatigue is strain and not stress controlled

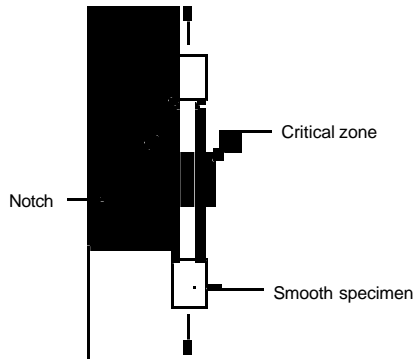
Taking the simple case of a plate with a hole loaded in tension (Figure 33) we see that although the material remote from the hole stays elastic the material in the most highly stressed region yields. The plastic deformation associated with this yielding will be the primary cause of fatigue, and following its accumulation as the loading history proceeds is likely to give an accurate life prediction.



*Figure 33. Plate with hole loaded in tension*

In the most common situation yielding only occurs in a small region at the root of a notch. The surrounding region remains elastic, and when the applied load is removed the yielded zone returns to its former shape. This means that the material does not just return to zero stress but is forced back to its former shape. Visualizing this situation as in Figure 34 it means that we must consider the behaviour of

specimens loaded in a “fixed grip” manner, so that strain and not stress is controlled. Deriving a strain-time history is then more difficult than obtaining a stress-time history for the elastic case.



*Figure 34. The concept of similitude between a strain controlled test specimen and a component being designed*

The Strain-Life method assumes similitude between the material in a smooth specimen tested under strain control and the material at the root of a notch. For a given loading sequence it is assumed that both will fail at the same time, as long as all other conditions are the same.

So why then, if fatigue damage is strain and not stress controlled, has the Stress-Life approach worked reasonably well in so many cases? In fact, the only reason that the Stress-Life approach worked at all is that, in the high-cycle regime, stress and strain levels are low and so stress and strain are almost linearly related. However, as the load levels become larger, the method breaks down and a more generalised, strain-based approach, which accounts for plasticity, must be adopted. This type of behaviour has been commonly referred to as low-cycle fatigue or more recently strain-controlled fatigue. The transition from low-cycle to high-cycle fatigue behaviour generally occurs in the range  $10^4$  to  $10^5$  cycles.

The local stress-strain history must be determined, either by analytical or experimental means. Stress analysis procedures such as finite element modelling, or experimental strain measurements are usually required.

A full understanding of the procedure needs a rather detailed description of how these local stresses and strains are found. This will be covered in section 5.1, but it is possible to use the methods without such detailed knowledge, and readers may wish to move directly to section 5.2. The minimum needs for basic understanding may be summarised as:

(i) Strain is the basic cause of fatigue. At some point in the component being loaded the strain must be plastic (i.e. non-reversible) for a crack to start.

(ii) Plastic strain changes the properties of a material, especially the way it yields when loaded later. Estimates of local stress-strain histories must allow for this, usually by using data that applies to material that has already been fatigued.

(iii) When a local stress-strain history has been calculated, a valid cycle-counting method must be used to identify ranges of strain that are being applied.

(iv) When strain cycles have been counted, experimental data from tests carried out under constant-strain conditions will be needed for the life estimation (not S-N data from conventional constant-load tests).

Point (ii) must be particularly kept in mind when using proprietary software. At some point the user will be asked to nominate a file containing material properties. These must be ones that have been measured in tests that applied repeated cycles of strain, called the cyclic material properties. The ordinary tensile test in which a single specimen is subjected to a steadily increasing load (a monotonic test) does not provide suitable data.

## **5.1 The Stress-Strain Curve**

### **5.1.1 Elastic and Plastic Strain.**

A typical loading sequence being analysed will consist of a series of peaks and troughs of such a magnitude that yielding occurs at each one. Reversed plastic strain is then the basic phenomenon. To quantify this we need

(i). A formula for the shape of the stress-strain curve, extending beyond yield

(ii). A way of estimating how much of the total strain is plastic.

Figure 35 shows a specimen loaded beyond yield and then unloaded. To satisfy (i) we need a description of the line OAB. To satisfy (ii) we need to quantify OC. It is important to note that these are strictly separate requirements. If the unloading line followed BAO we would be dealing with non-linear elasticity, not plasticity. The characteristic of plasticity is that the dimensions of a region have been changed. The simplest way of specifying this change is to consider the unloaded condition.

By co-incidence, though, the conditions to satisfy (i) and (ii) are the same. Tests show that the line BC is always parallel to OA. This means that the plastic strain at



B is OC, which is precisely the component contributed by curvature. A formula for the curved stress-strain line then consists of two parts, a linear one and a curved one, and the curved component can be called plastic strain.

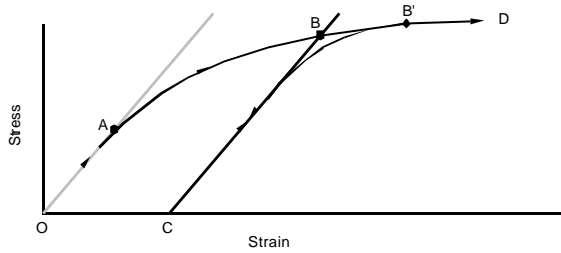


Figure 35. Categorising the true stress-strain curve

Looking first at OA, this is a straight line with a slope given by the modulus of elasticity,  $E$ , so

$$E = \frac{\mathbf{S}}{\mathbf{e}} \quad \text{and} \quad \mathbf{e}_e = \frac{\mathbf{S}}{E}$$

In the region AB curvature causes a greater strain for a given stress. Because this curvature is a measure of the plastic strain we can express it in terms of plastic strain:

$$\mathbf{s} = K(\mathbf{e}_p)^n$$

where:

$K$  the constant of proportionality known as the **strength coefficient**.

$n$  is known as the **work hardening exponent**.

$\mathbf{e}_p$  is the **plastic strain**.

The plastic strain,  $\mathbf{e}_p$ , can now be written as:

$$\mathbf{e}_p = \left[ \frac{\mathbf{S}}{K} \right]^{\frac{1}{n}}$$

The total strain is then the sum of elastic and plastic strains, giving:-

$$\mathbf{e}_t = \frac{\mathbf{S}}{E} + \left[ \frac{\mathbf{S}}{K} \right]^{\frac{1}{n}}$$

This expression is sometimes called the Ramberg-Osgood relationship.

### 5.1.2 Cyclic Stress-Strain Behaviour

If from point C in Figure 35 the loading direction is again reversed (i.e., the specimen is loaded back towards point B and beyond), the stress-strain curve will follow the trace C-B'-D. As in the case of unloading, most of the trace will be a straight line with slope equal to the modulus of elasticity,  $E$ . Once beyond B' the material has apparently “forgotten” the excursion B-C-B' and goes on to fracture at D as though it had never happened. The excursion B-C-B' is therefore only an interruption in the primary loading process O-A-B-D, and can be treated as a fatigue cycle. These observations are central to the local Strain-Life methodology.

Figure 36 shows another possible development. After loading from O to B and unloading to C deformation is continued into the negative strain region. At first the line continues straight, but at D compressive yielding begins and the line curves again. If the loading direction is again reversed at a compressive stress of  $-S_{max}$  and the specimen is loaded back to  $+S_{max}$  a complete loop will be defined). The loop BCDFB is called a hysteresis loop and defines, in stress-strain space, a single fatigue cycle.

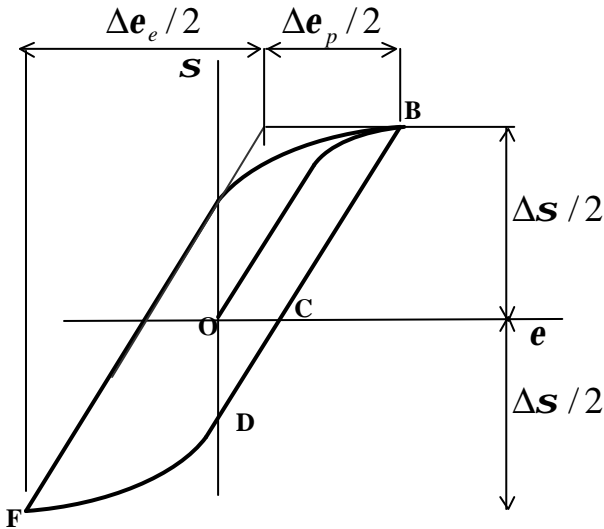


Figure 36. Cyclic stress-strain behaviour

The total strain range is then made up of the elastic and plastic components:

$$\Delta \mathbf{e}_t = \Delta \mathbf{e}_e + \Delta \mathbf{e}_p$$

which may be written as:

$$\Delta \mathbf{e}_t = \frac{\Delta \mathbf{s}}{E} + \Delta \mathbf{e}_p$$

### 5.1.3 Cyclic Loading Under Strain Control

If a material is repeatedly cycled between fixed strain limits one of several things may happen depending on its nature and initial conditions of heat treatment. The behaviour will affect the shape of the hysteresis loop. If the yield stress rises as more cycles are applied the loop heights will increase in size and the material is said to work harden. The opposite may happen, with loops decreasing in size as the material work softens. Sometimes neither work hardening nor work softening takes place. It is important to allow for these changes when the strain-time history is being computed. It would be possible to do this computation using current values for the stress-strain data, modified after every loop. Apart from the considerable increase in computing time which this would need, handling the material property data would cause substantial difficulty. The compromise which has been reached is to use properties which are the *average* values likely to occur during the life. There are a number of ways of measuring these when conducting strain-controlled tests but the important thing for a user of Strain-Life software are that these are listed as cyclic properties, not monotonic ones. To distinguish between the cases a prime is used for cyclic properties such as  $K'$  and  $n'$  rather than  $K$  and  $n$  for monotonic. We then have

$$\mathbf{e}_t = \frac{\mathbf{s}}{E} + \left[ \frac{\mathbf{s}}{K'} \right]^{1/n'}$$

where,

$K'$  is called the *cyclic strength coefficient*.

$n'$  is called the *cyclic strain hardening exponent*.

It must be emphasised that it is not possible to carry out valid Strain-Life calculations without a database of cyclic properties

### 5.1.4 Hysteresis Loop Shape

When constructing a strain-time history from stress-time data hysteresis loops have to be constructed using the cyclic stress-strain curve. One key step is to determine

when the line begins to curve as we move from tension to compression and vice versa. This is handled by using Masing's Hypothesis, which assumes that the line describing a stress-strain hysteresis loop is geometrically similar to the cyclic stress strain curve but numerically twice its size. Consequently, the equation for the curve can be derived directly from the equation of the cyclic stress-strain curve.

Consider any point on the cyclic stress-strain curve with coordinates  $(\mathbf{e}_1, \mathbf{s}_1)$ . It follows that:

$$\mathbf{e}_1 = \frac{\mathbf{s}_1}{E} + \left[ \frac{\mathbf{s}_1}{K'} \right]^{1/n'}$$

From Masing's hypothesis, the same point can be located on the hysteresis loop curve and it will have coordinates  $(\Delta\mathbf{e}_1, \Delta\mathbf{s}_1)$  where:

$$\begin{aligned} \Delta\mathbf{e}_1 &= 2\mathbf{e}_1 \\ \Delta\mathbf{s}_1 &= 2\mathbf{s}_1 \end{aligned}$$

Substituting into the equation for the cyclic stress-strain curve:

$$\frac{\Delta\mathbf{e}_1}{2} = \frac{\Delta\mathbf{s}_1}{2E} + \left( \frac{\Delta\mathbf{s}_1}{2K'} \right)^{1/n'}$$

which in the general case reduces to:

$$\Delta\mathbf{e} = \frac{\Delta\mathbf{s}}{E} + 2 \left( \frac{\Delta\mathbf{s}}{2K'} \right)^{1/n'}$$

The relationship between the cyclic stress-strain curve and hysteresis loop shape is illustrated in more detail in the following example.

### 5.1.5 Example: Calculating Cyclic Stress-Strain Response

A test piece with the following cyclic properties,

cyclic strength coefficient,  $K' = 1200 \text{ MPa}$   
 cyclic strain hardening exponent,  $n' = 0.2$

modulus of elasticity,

$$E = 210,000 \text{ MPa}$$

is to be cycled under strain control with a fully-reversed strain range of 0.03.

In calculating the stress-strain response, it will be necessary to make two assumptions.

Firstly, it will be assumed that the material will follow the cyclic stress-strain response rather than the monotonic one on initial loading from zero to the maximum strain amplitude of 0.015. While there is a case for the use of the monotonic response for the first half cycle, in practice after repeated loading, the resulting hysteresis loop tip always migrates towards the cyclic stress-strain curve.

Secondly, it will be assumed that the material will exhibit cyclically stable response from the initial loading. A more precise analysis would require accounting for the cyclic hardening or softening characteristics of the material. As in the case of the first assumption above, the cyclic modelling process is used to model the stable hysteresis behaviour since in most cases this occupies the majority of fatigue life.

Figure 37(a) illustrates a portion of the strain time sequence that is to be used to cycle the test specimen. The value of the stress corresponding to the first turning point, 1, can be calculated directly from the equation for the cyclic stress-strain curve,

$$\epsilon_1 = \frac{\mathbf{S}_1}{E} + \left[ \frac{\mathbf{S}_1}{K'} \right]^{\frac{1}{n'}}$$
$$0.015 = \frac{\mathbf{S}_1}{210,000} + \left[ \frac{\mathbf{S}_1}{1200} \right]^{\frac{1}{0.2}}$$

The value of  $\mathbf{S}_1$  can be calculated from the above equation by using an iterative technique such as Newton Raphson or interval halving. Such a procedure will provide a value of

$$\mathbf{S}_1 = 500 \text{ MPa}$$

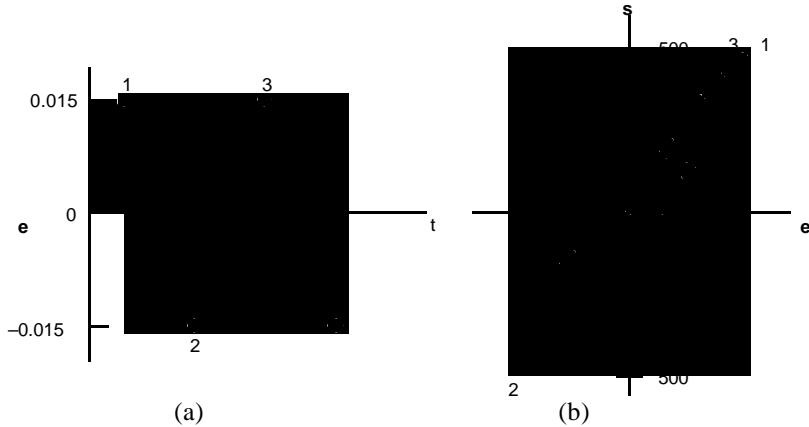


Figure 37. Strain history (a), and corresponding stress-strain response curve (b)

The value of the stress at the next turning point, 2 (Figure 37 (a)), can be calculated by considering the total strain range, 0.03, together with the equation for twice the cyclic stress-strain curve,

$$\Delta \mathbf{e} = \frac{\Delta \mathbf{s}}{E} + 2 \left( \frac{\Delta \mathbf{s}}{2K'} \right)^{\frac{1}{n'}}$$

$$0.03 = \frac{\Delta \mathbf{s}}{210,000} + 2 \left( \frac{\Delta \mathbf{s}}{2400} \right)^{\frac{1}{0.2}}$$

Solving this equation by iteration leads to a value of  $\Delta \mathbf{s} = 1000$  MPa. The co-ordinates of the second turning point in stress-strain space can now be calculated from:

$$\mathbf{e}_2 = \mathbf{e}_1 - \Delta \mathbf{e} = (0.015 - 0.03) = -0.015$$

$$\mathbf{s}_2 = \mathbf{s}_1 - \Delta \mathbf{s} = (500 - 1000) = -500 \text{ MPa}$$

The co-ordinates of the third turning point can be calculated in the same way as for the second point, and for the fully reversed loading being considered here, they must be the same as for the first turning point. Figure 37(b) illustrates the stress-strain hysteresis loop corresponding to the induced cyclic loading

### 5.1.6 Estimating a Strain-Time History.

Example 5.1.5 was concerned with calculating the hysteresis loop for one cycle of loading. If the stress cycle is repeated the loop will also be repeated if the material has reached a stable condition. When constant amplitude stress cycles are being applied the strain-time history will be just a series of repetitions of the strain pattern determined by the hysteresis loop. This is the history needed to make a prediction from tests conducted at constant strain. This step is not needed in the Stress-Life or S-N approach. In that case the loading conditions are in terms of  $S$  and the material data is also in terms of  $S$ .

Normally the loading history will be variable amplitude, and the time history will have been reduced to a sequence of peaks and troughs in stress. This must be transformed into a series of peaks and troughs of strain. This will be followed by a cycle counting step, a cumulative damage hypothesis and damage estimation using Strain-Life data. The three methods that have been used for the transformation are:

(i) Test a specimen with a strain gauge at the notch root. This gives a direct link between load and local strain, which can be used in calculation. The test may be monotonic, but to be valid the material should have reached a stable state. This means it has to be plastically cycled before measurements are made, probably before the strain gauge is applied.

(ii) Use a stress analysis method that allows for plasticity to calculate the curve linking local strain with nominal load. This only has to be done once, for the greatest range of nominal load. Fatigue calculations can then take this as a “look-up” table to follow the service history.

(iii) Use an empirical relationship linking stress and strain at the notch root to the nominal load. This cannot be a simple Hooke's Law formula, since it has to be valid in the plastic region.

The laboratory tests needed for method (i) are too clumsy for routine design predictions. Method (ii) normally uses Finite Element Analysis and could be expected to be widely used in modern conditions, but this is not the case. In spite of major advances in computer-based stress analysis method (iii) still dominates design software.

Only one empirical formula has ever been seriously, and extensively, used in this field. It is known as “Neuber's Rule”. The fundamental point is to separate stress and strain by having two 'concentration' factors instead of one. In the elastic region  $K_T$  controls both stress and strain. This breaks down once strain stops being proportional to stress. Neuber suggested having different concentration factors for stress and for strain. Before giving a formal definition of these it is worthwhile trying to imagine what will happen when the stress-strain plot starts to curve. A

given increase in stress will cause a larger strain increase after the line starts to curve because curvature always flattens the line towards the strain axis. The ratio (greatest-stress)/(nominal-stress) in the component will start to fall as the load climbs above the value which has first caused yielding at the notch root, whereas the ratio (greatest-strain)/(nominal-strain) will rise. The two ratios may be defined as:

$$K_s = \frac{\text{Maximum stress at the notch}}{\text{Nom stress remote from the notch}}$$

and

$$K_e = \frac{\text{Maximum strain at the notch}}{\text{Nom strain remote from the notch}}$$

Until yielding starts at the notch root  $K_s = K_e$  and both are equal to  $K_T$ . After that  $K_s$  decreases and  $K_e$  increases. Neuber's Hypothesis assumes that the product  $K_s \cdot K_e$  stays constant. If yielding occurs at the notch root but not in the material away from the notch we can use:

$S$  = Nominal stress remote from the notch

$e$  = Nominal strain remote from the notch

$\mathbf{S}$  = Local stress at the notch root

$\mathbf{e}$  = Local strain at the notch root

Then after yielding the *strain* concentration factor,  $K_e = \frac{\mathbf{e}}{e}$  and is  $> K_T$

and the *stress* concentration factor,  $K_s = \frac{\mathbf{S}}{S}$  and is  $< K_T$

Before yielding both are equal to  $K_T$ , so Neuber's Hypothesis leads to:

$$K_e \cdot K_s = (K_T)^2$$

Re-arrangement of this gives the useful equation:

$$(K_T)^2 S e = \mathbf{s e}$$

Another re-arrangement gives:

$$(K_T e)^2 E = \mathbf{s e}$$



Parameters on the left side can be computed by FEA, and establish point 2 in Figure 38. The target is then to find a point on the cyclic stress-strain ( $\sigma$ - $\epsilon$ ) curve which gives the same product. Numerical iteration is used to find point 3 in Figure 38.

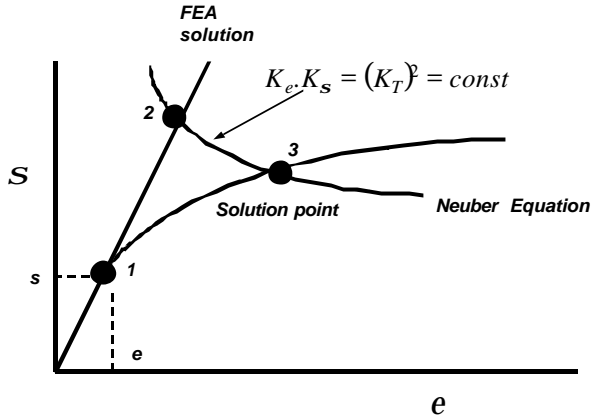


Figure 38. Graphical solution using Neuber's Rule

This process is carried out for each transition from peak to trough or trough to peak in the loading history, called a reversal. Note that account must be taken of the changes in slope where the trace rejoins a previous point in the history, a phenomenon shown in Figure 39.

This calculation results in a pattern of nested hysteresis loops like those shown in Figure 39. From these loops we can estimate a strain-time history (shown with the time axis vertical) and extract cycles for life estimation.

If we make the very plausible assumption that the fatigue damage depends on the size of the minor loops in the stress-strain curve a strain cycle will be an event like 3-4, 8-10 or 2-5. These are loops that would be identified by the Rainflow technique described in Figure 39. In the Strain-Life approach, then, we can place a physical meaning on Rainflow counting.

### 5.1.7 Retaining Mean Stress Information

One point about the cycles extracted in Figure 39 is that ones with identical strain ranges may have different stress ranges. Cycle 3-4 and 6-7, for instance, have the same range but different mean stresses. The data may therefore be retained either as a single dimension probability distribution or a two-dimensional range/mean

matrix. Mean stress effects are sometimes ignored, but if they are to be allowed for the two-dimensional form is essential.

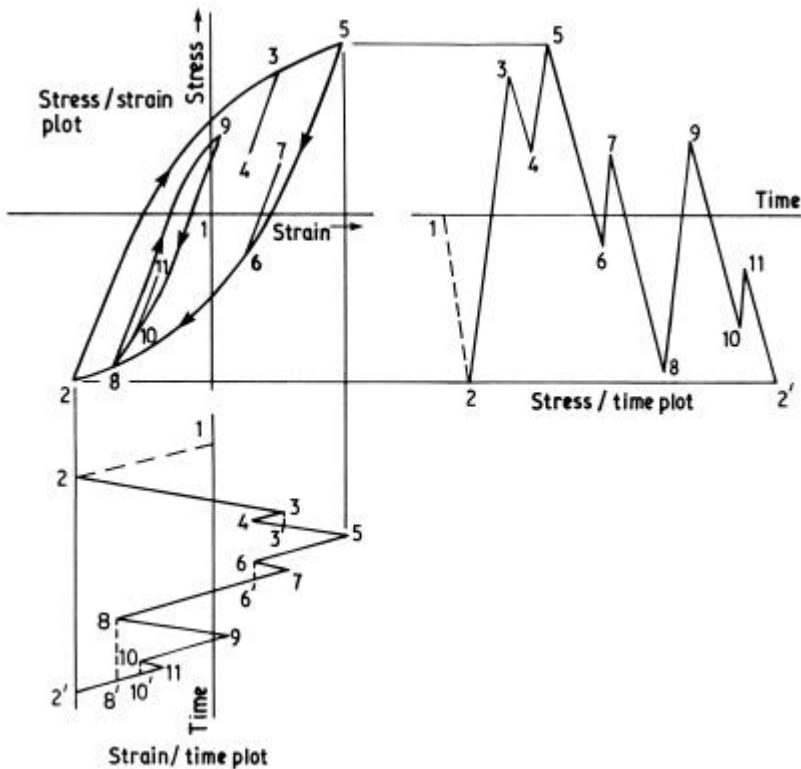


Figure 39. Hysteresis loops of local stress and strain under a short random sequence

## 5.2 The Strain-Life Curve

Once the number and magnitude of strain reversals in a time history have been established experimental data about life in reversals and plastic strain magnitude is needed. Smooth specimens with an 'hour-glass' shape designed so that they do not buckle in compression are taken through tension-compression cycles of constant strain amplitude. Mean strain is usually zero. A clip-on extensometer is kept in place throughout the test so that strain at the specimen can be properly controlled. It would be possible to record only two values for each test, that is the total strain amplitude and the number of reversals needed to cause complete failure of the specimen. With a number of specimens tested at different strain amplitudes the

Strain-Life plot would then be similar to a Stress-Life  $S$ - $N$  line. A more refined approach is normal, though. Periodically during the test a record may be taken of the stress-strain loop. It will look like Figure 36. The plastic part of the total strain is by definition the horizontal distance between the two points where the graph crosses the line at zero stress,  $\Delta \epsilon_p$ . The elastic component  $\Delta \epsilon_e$  is then the difference between total strain and plastic strain, and a value of Young's Modulus can be calculated. The separate contributions of elastic and plastic strain may then be allowed for in the final data reduction.

When considering the S-N approach the relationship between  $S$  and  $N$  is taken as

$$N = aS^{-b}$$

where  $N$  is the number of cycles to failure and  $S$  is either the amplitude or the range of stress. This is sometimes known as the *Basquin* equation. In the Strain-Life method amplitude of stress is normally used, with the symbol  $\Delta \sigma$ . Because of the way the strain-time history is constructed it is rational to use a reversal between a maximum and a minimum as a life unit, with a symbol  $N_f$ , although the usual Rainflow counting algorithm always pairs all reversals into cycles. To convert the S-N relationship for use with the elastic component of strain we therefore have  $2N_f = a(\sigma_a)^{-b}$ .

If  $\sigma_a = \sigma'_f$  when  $2N_f = 1$  then  $1 = a(\sigma'_f)^{-b}$  and "a" can be eliminated, giving

$$S_a = S'_f (2N_f)^b$$

where:

$S_a$  is the true cyclic stress amplitude

$S'_f$  is the  $\sigma_a$  intercept at  $2N_f = 1$ , called the *fatigue strength coefficient*

$2N_f$  number of half cycles, *reversals*, to failure

$b$  is the slope called the *fatigue strength exponent*. Note that this  $b$  is a different  $b$  from the one used in the Stress-Life approach.

$S'_f$  and  $b$  are considered to be material properties usually found by regression analysis of the experimental data.  $S'_f$  is approximately equal to the monotonic true fracture stress,  $S_f$ , and  $b$  varies between -0.05 and -0.12.

The equation may be rewritten in terms of elastic strain amplitude:

$$\mathbf{e}_e = \frac{\mathbf{S}_a}{E} = \frac{\mathbf{S}'_f}{E} (2N_f)^b$$

where:

$\mathbf{e}_e$  is the elastic strain amplitude  
 $E$  is the modulus of elasticity

Next we need to include the effect of the plastic element, whose effect it strongest when failure occurs at a low number of cycles. This is handled by the Manson-Coffin expression that relates the plastic strain component of a fatigue cycle to life by a simple power law:

$$\mathbf{e}_p = \mathbf{e}'_f (2N_f)^c$$

where:

$\mathbf{e}_p$  is the plastic strain amplitude  
 $\mathbf{e}'_f$  is the regression intercept called *fatigue ductility coefficient*  
 $2N_f$  is the number of reversals to failure  
 $c$  is the regression slope called the *fatigue ductility exponent*

$\mathbf{e}'_f$  and  $c$  are considered to be material properties.  $\mathbf{e}'_f$  is approximately equal to the monotonic fracture strain,  $\mathbf{e}_f$ , and  $c$  varies between -0.5 and -0.8.

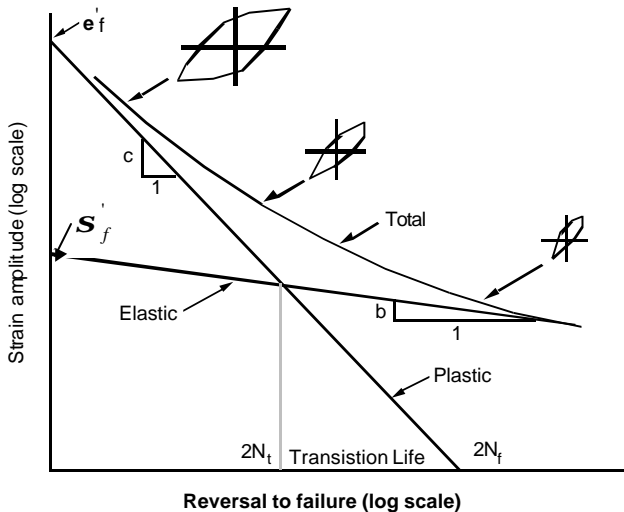


Figure 40. The total Strain-Life curve

Mathematically, this curve can be described by summing the Basquin and the Manson-Coffin component curves:

Then:-

$$\mathbf{e}_t = \mathbf{e}_e + \mathbf{e}_p$$

$$\mathbf{e}_t = \frac{\mathbf{S}'_f}{E} (2N_f)^b + \mathbf{e}'_f (2N_f)^c$$

Note that there is no fatigue limit defined by this equation and so it is usual to define a *cutoff* in terms of reversals that specifies an endurance limit. Typically, the cutoff is set to about  $5 \times 10^7$  reversals.

### 5.2.1 The Effect of Mean Stress

Mean stresses have to be accounted for in a similar way to the Stress-Life approach. A number of approaches have been proposed including ones by Morrow and by Smith-Watson-Topper. However, only the Morrow correction is detailed here.

### 5.2.2 The Morrow Mean Stress Correction

Morrow was the first to propose a modification for mean stress effects by modifying the baseline Strain-Life curve. He suggested that mean stress could be taken into account by modifying the elastic part of the Strain-Life curve by the mean stress,  $\mathbf{S}_0$ .

$$\mathbf{e}_e = \frac{(\mathbf{S}'_f - \mathbf{S}_0)}{E} (2N_f)^b$$

the entire strain life curve then becomes:

$$\mathbf{e}_a = \frac{(\mathbf{S}'_f - \mathbf{S}_0)}{E} (2N_f)^b + \mathbf{e}'_f (2N_f)^c$$

The Morrow equation is consistent with the observation that mean stress effects are significant at low values of plastic strain and have little effect at high plastic strains.

## **5.3 FE Hints and Tips – e-N Approach**

### **5.3.1 Element vs Nodal Results**

The same issues arise concerning the choice of element v nodal results and whether or not to average as was done with the Stress-Life approach as detailed in section 4.9.2.

### **5.3.2 Weld Analysis, Component Material Curves and Composites**

When calculating stresses and strains for a fatigue life analysis with FE models we generally use linear elastic analysis. These local stresses and strains are then converted using, for instance, Neubers Rule, into approximate local values. Because of this it is very important to realise that certain types of fatigue calculations are not valid with the Strain-Life approach. Weld analysis is a subset of the component S-N approach in that a reference (or nominal) stress value is used to cross reference to a component S-N material curve for that same configuration. We therefore have no basis on which to perform local stress and strain conversions since the reference stress used is generally away from the critical region. Similar comments also apply to composites where the Stress-Life approach is also generally used instead of the Strain-Life approach.



# 6 Crack Propagation Analysis Using LEFM

## 6.1 How Fatigue Cracks Start

It is now well accepted that, in general, fatigue cracks start following the nucleation of a crack on a *persistent slip band* (PSB). After this initiation there are usually two stages of stable crack growth before a third stage of rapid growth causes failure. The initial stage is concerned with the propagation of small cracks over a few grain boundaries and typically takes up little of the overall fatigue life. This phase of crack growth is dominated by shear cracking and is known as Stage I. After traversing one or two grains a change in crack mode is observed, usually but not always at a grain boundary. This is readily observed in a push-pull fatigue specimen, since the crack turns through a 45° angle to propagate by a new mechanism on a plane normal to the maximum principal stress. Such growth is essentially non-crystallographic, although microstructural features do have influence when grains are favourably oriented to provide an easy or rapid crack growth path. This phase is called Stage II

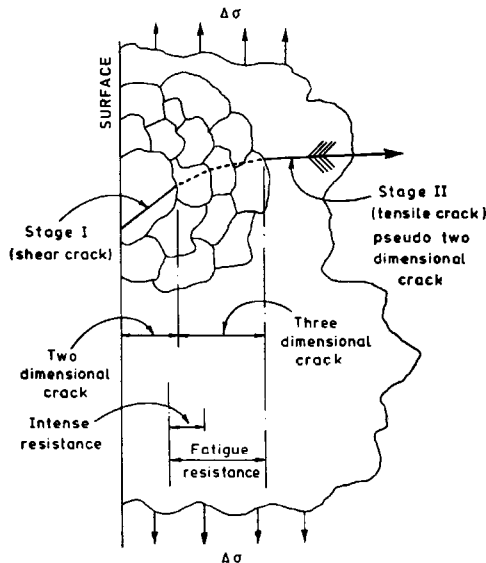


Figure 41. The transition from stage I to stage II crack growth



When a certain critical condition is reached a marked increase in growth rate occurs. Progress to catastrophic failure then occurs in only a few more cycles, and this phase is normally ignored in design.

## 6.2 The Concept of Stress Intensity

Modern calculations use the concepts of Fracture Mechanics. These were originally developed to compute the static loads that can be carried by components containing cracks, before being extended to cover fatigue crack propagation. The difficulties can be illustrated by referring to Figure 42. A circular hole in a plate carrying uniaxial tension has a  $K_T$  of 3, which increases if the hole is elongated in a direction perpendicular to the principal stress. A possible model of a crack is an ellipse like this but with finite length  $a$  and zero width  $b$ . The simple formula given in the figure then predicts that  $K_T$  will be infinite, so that any value of the nominal stress  $\sigma$  will cause an infinitely high tension at the crack tip. This implies that the crack will propagate catastrophically by tearing. Experiments show that this does not happen. Cracked bodies will carry some load, so the analysis must be faulty and an alternative is needed.

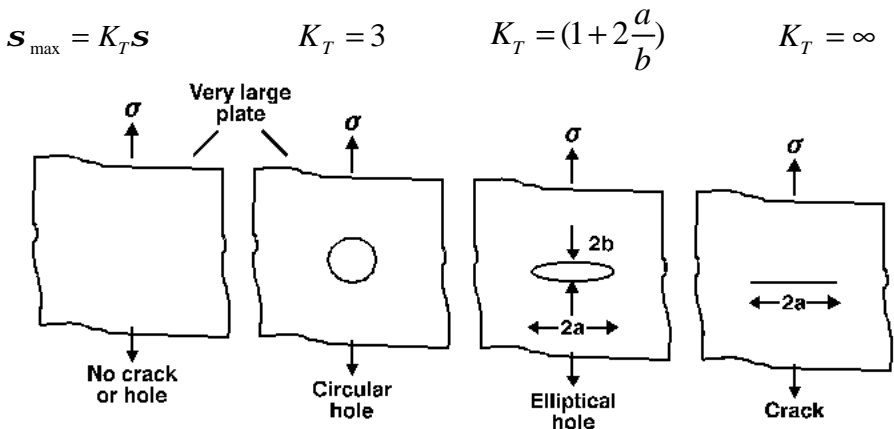


Figure 42. Elastic stress concentrations

Early theories concentrated on the energy absorbed when the crack moves forward, but we now concentrate on the stress field around the tip of the crack. Westergaard showed that this field can be expressed in such a way that all terms (normal stresses and shear stresses on all elements, Figure 43) can have a common factor  $\sigma(\pi a)^{1/2}$  extracted. Geometry, form of loading, and the way the crack will extend also have effects that it may be possible to quantify for the whole field. Combining these into a factor  $Y$  we can identify a parameter that characterises the whole stress

field near to the crack tip. Calling this the *crack tip stress intensity factor*,  $K$ , we have:-

$$K = YS(\rho a)^{1/2}$$

If the crack extends when  $K$  reaches a critical value we will have a criterion for determining the stress  $\sigma$  which will extend a crack of length  $a$  when  $Y$  has a certain value. Tests show that this is the case.

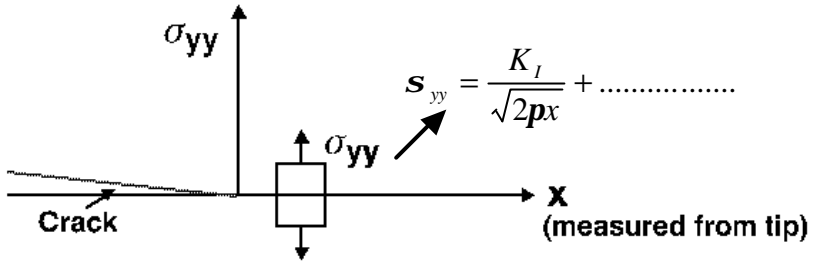


Figure 43. The concept of stress intensity

Continuing to limit the discussion to the case of static loading, the critical value that  $K$  must reach is a material property called *fracture toughness*, symbol  $K_{Ic}$ . Analysis is often limited to the case where the plastic zone near to the crack tip is small (Figure 44), and the topic is then called Linear Elastic Fracture Mechanics (LEFM). The three modes of extension that can take place are identified as Mode I, Mode II and Mode III as shown in Figure 45 and solutions for  $Y$  will depend on the mode as well as on component geometry. Closed-form solutions for  $Y$  exist in many cases, and numerical solutions using FEA are common. Several compendia of expressions have been published such as the well known Rooke and Cartwright publication.

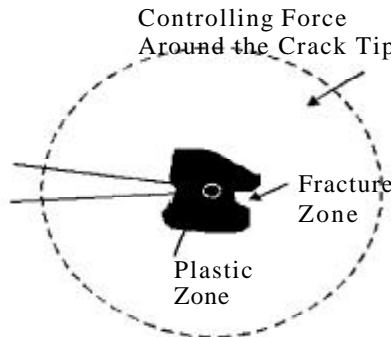


Figure 44. Crack tip yielding

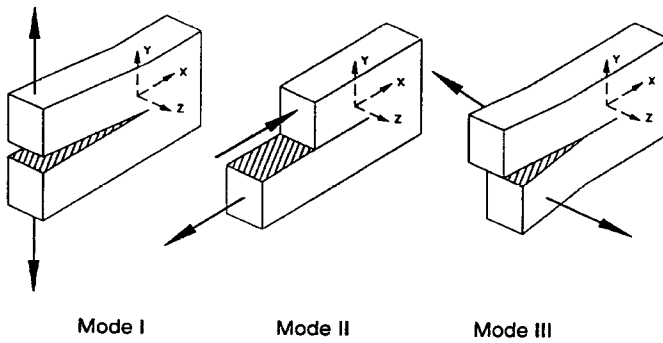


Figure 45. The 3 modes of cracking used in fracture mechanics

### 6.3 Fatigue Crack-Propagation and LEFM

Static LEFM provides a numerate criterion for the onset of catastrophic failure in a fatigue test. If a crack is propagating under repeated loading  $a$  will steadily increase and the value of  $K$  when the dynamic load is a maximum will also increase. When this value,  $K_{Max}$ , reaches  $K_{Ic}$ , a material property, the crack will extend catastrophically. LEFM contributes more than this to fatigue, though. In a constant amplitude fatigue test ranging from  $\sigma_{Max}$  to  $\sigma_{Min}$  values of  $K$  can be computed for the extremes of load, say  $K_{Max}$  and  $K_{Min}$ . The difference between these is the range of crack tip intensity factor, say  $\Delta K$ . As the rate of crack propagation  $da/dN$  depends on  $\Delta K$  its value can be calculated. The rate will depend on  $\sigma$  and on crack length  $a$ , which will increase as the crack extends. To carry out tests at constant  $\Delta K$  crack length must be monitored and the range of load decreased in proportion to  $a^{1/2}$ .

Figure 46 shows typical results for tests carried out at constant  $\Delta K$ . There are three regions. Over the middle range of  $\Delta K$  the relationship is linear. Both scales are logarithmic, which means that an expression for this middle portion is:

$$da/dN = C(\Delta K)^m$$

where  $da/dN$  = current rate of crack propagation,  
 $a$  = current crack length, and  $C, m$  are material properties.

This is the most-used expression, called the **Paris-Erdogan** equation. For high values of  $\Delta K$  the graph diverges from a straight line. This is not particularly

important in most practical cases because the crack is growing so rapidly that big variations in the formula used here make little difference to the predicted life. Much more important is the deviation from the straight line at low  $\Delta K$  values. If the line becomes vertical, as it does in many reported data, there is a **threshold** value of  $\Delta K$  below which the crack growth rate is zero.

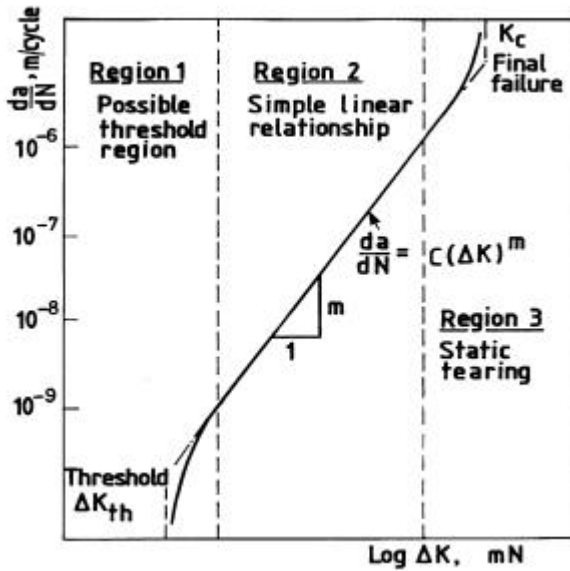


Figure 46. Crack propagation rates at constant  $\Delta K$

The majority of fatigue crack propagation studies have been associated with long cracks (i.e. of order 10 mm) and low cyclic stresses, which are well described by linear elastic fracture mechanics. Since Mode I cracking is the most easily studied, using standard specimen geometries, Mode II and Mode III crack growth data are scarce. Mode I is the most common type of in-service failure.

The curve in Figure 46 is sometimes called the **apparent**  $\Delta K$  curve. There are many effects that the Paris equation does not take into account, such as crack closure, corrosive environments, the influences of a notch, and static fracture mode contributions. One way to model these effects is to derive an **effective**  $\Delta K$  curve modifying the apparent  $\Delta K$  curve through all three of its regions. This effective  $\Delta K$  is then used in the Paris equation to determine crack growth

## 6.4 Stress Intensity Factor $K$ Versus Compliance-Function $Y$

The factor  $Y$  is sometimes called the **Compliance-Function**. In physical terms it is the change in stiffness or flexibility as the crack grows, i.e., the structure becomes more compliant as the crack gets longer.  $Y$  is unity for a central through-thickness crack in an infinite body and 1.12 for an edge crack. For more realistic geometries, the Compliance-Function,  $Y$ , is normally more complex. Many hundreds of  $K$  solutions are available. The designer's task is not to derive new expressions for  $Y$  but to identify the geometry and loading present and find an existing formula.

The relationship between stress intensity, stress, and crack length is known as the **fracture mechanics triangle** (Figure 47). If two of the corners are known the other can be derived.

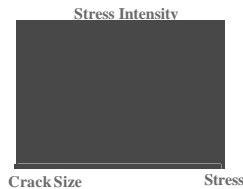


Figure 47. The fracture mechanics triangle

The fracture mechanics triangle (Figure 47) allows a designer to compute any one of the three parameters given the other two. Using  $\Delta K$  and the Paris equation the instantaneous rate of crack propagation at a given value of  $a$  can then be calculated. An extension of this, using some form of integration, will give the number of cycles needed to propagate a crack from an initial to a final length. This leads to the fatigue crack propagation rectangle of Figure 48, where knowing any three corners allows the fourth to be computed. When considering crack growth, then, there is a relationship between stress range and life just as there is with the Stress-Life (S-N) method. In crack growth, though, life is closely related to initial and final crack lengths. This forms the basis of a life estimation method that underlies the Damage-Tolerance philosophy described earlier.

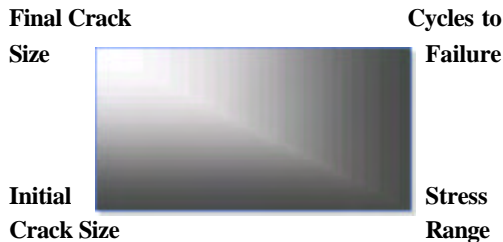


Figure 48. The fatigue crack propagation rectangle

## 6.5 What is the Meaning of “Nominal Stress” as Used With a Compliance-Function in FE Based Crack Growth Calculations?

The definition of *nominal* stress, for crack growth calculations, is similar to that used in component S-N curves. Rather than try to consider the stress in the region of the crack tip, the stress conditions around the crack tip are characterised using  $K$ , the stress intensity factor. This value is a function of the nominal stress (from the uncracked body) and the crack length. The value of  $K$  is further modified by the  $K$ -solution, or Compliance-Function, this is a function of crack ratio. The crack ratio varies from 0 (uncracked) to 1 (fully cracked). This Compliance takes into account two effects. Firstly as the crack grows the stresses may re-distribute and secondly the crack may be growing into higher or lower-stressed areas. So it is very important to use the correct “nominal” stress.

Consider the example shown in Figure 49.

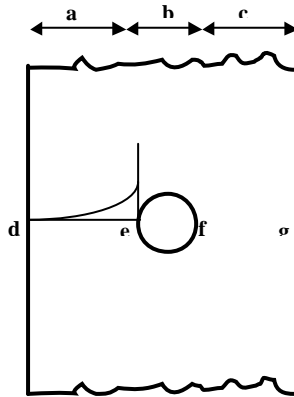


Figure 49. Compliance-Functions for a hole in a plate

Hypothetically it is possible to equate the following two calculations.

- [1]. Use a Compliance-Function for the hole and the nominal stress in section  $d-e$ . This is the stress if the hole was not there, ie, half total load divided by area  $d-e$ .
- [2]. Use a Compliance-Function for an edge crack in an infinite plate and the true stress at  $e$ . In this case the nominal stress is the true stress at  $e$ .

Although in [2] a stress approximately 3 times bigger will be used in the calculation, the resultant crack growth rates should be similar because the different Compliance-Functions for [1] and [2] will account for the different stress levels.

There are many practical problems with this hypothetical situation, one of which is the redistribution of load as one side of the hole cracks across the section. It does, however, highlight the difficult task of identifying, and obtaining, the nominal stress values.

## **6.6 What is the Role of FEA in Generating Compliance-Functions?**

If a suitable Compliance-Function can be found in the literature then it will probably provide the fastest solution to the problem. The alternative is to generate a  $K$  solution using FE. This will be specific to the exact geometry and loading conditions encountered. Such a Compliance-Function can be generated by introducing a small crack into the FE model, using crack-tip elements. These often have the advantage of using energy based methods which are more accurate and quicker to converge than stresses. The engineer then analyses the model and studies the principal stress directions. Some FE systems will indicate the direction of crack growth that gives the maximum energy release. The engineer would be wise to refer to other NAFEMS publications on this subject, including the LEFM benchmarks. The crack should then be 'grown' in the appropriate direction in modest steps (perhaps 10 from initial crack to fully cracked).

Some systems may automate this process. This leads to a table of results of  $K$  versus crack size which may be directly useful, for example demonstrating leak-before-burst. But if a fatigue life is required then they need to be used in crack propagation rate calculations, such as the Paris-expression. This set of results can be normalised into a  $K$  solution, dividing the crack size by section thickness (crack ratio) and normalising the value of  $K$  (from its equation) using the same reference stress for all cases.

It should be noted that the load used in the FE can be a unit loadcase, or a limit load case, whichever is more useful, since it is removed when the Compliance-Function is generated. If the fatigue loading is constant amplitude, the crack propagation calculations will be relatively straight forward. If it is variable, it is important that the applied loading is consistent with that used in the FE model, and uses a cycle by cycle growth to address each load application in turn.

## **6.7 Using Fracture Mechanics in Damage-Tolerant Design.**

In one version of Damage-Tolerant design the initial crack length  $a_i$  is determined by the longest crack likely to be missed at inspection, and the final crack length  $a_f$  is the one which would cause catastrophic failure. If the applied fatigue loading is constant-amplitude, the steps in the calculation are:

(i) Estimate the size of crack likely to be present in the component when it is first put into service,  $a_i$ .

(ii) From a measured value of  $K_{Ic}$ , estimate the maximum crack length,  $a_f$  which the component will tolerate when the applied stress reaches maximum tension. An expression for the crack-tip stress intensity factor will be needed.

(iii) Using the same expression for crack-tip stress intensity factor, calculate  $\Delta K$ . The simplest convention for the lowest value  $K_{min}$ , assumes that the crack closes when the load reaches zero, so that compressive stresses are ignored. The expression for  $\Delta K$  may depend on crack length, sometimes in a simple way but sometimes in a more complex way.

(iv) Substitute  $\Delta K$  into the Paris equation to obtain a crack propagation rate. This will put  $da/dN$  in terms of crack length  $a$ .

(v) Integrate this equation between  $a = a_i$  and  $a = a_f$  to give the number of cycles needed to grow a crack from  $a_i$  to  $a_f$ . This is the predicted life of the component. A classical integration is adequate if the Compliance Function is constant. Other Compliance Functions will require a numerical integration, as will any non-constant amplitude signal.

The procedure can vary from a very simple calculation to one needing a computer, according to the conditions. The results may be very accurate or rather dubious, depending on what guesses have to be made about the information used. Listing some of the problems and decision points:

(a) Estimation of the initial crack length is critical and difficult. Because the crack grows slowly at first, small changes here make big differences in the predicted life. Information from non-destructive tests (NDT) is usually proposed, but it is not likely that an expert in NDT will put a figure on the largest crack an inspection might miss unless a lot is known about the conditions. If a threshold  $\Delta K$  is known,  $a_i$  can be calculated from this, but then we are assuming that NDT can find any larger cracks.

(b) The crack is likely to start at the root of a notch, and its associated local concentration. Its early growth will then take it across a region with a stress gradient. We do not yet know how to modify the Paris equation to deal with this.

(c) There are still problems about putting a value on  $K_{min}$ . Cracks sometimes grow faster than the Paris equation predicts if the load goes into compression. In some



circumstances the crack does not close at zero load, so that part of the compression load is damaging. Working with an effective  $DK$  as suggested earlier may alleviate this.

(d) The equation for crack growth in terms of crack length may be difficult to integrate. Any serious work will be using proprietary software, though, and a numerical integration routine will be built in. Closed-form solutions to particular cases are of limited use.

If the loading is not constant amplitude the increment of  $a$  for each reversal is computed and added to the total length. Cycle-by-cycle algorithms like this are the fracture mechanics equivalent of the summations carried out in the Strain-Life method, but in Crack-Propagation a steady increase in ‘damage’ rate occurs as the crack gets longer (unless the crack is growing quickly into lower stresses areas). There are also known effects of sequence, such as the slowing down of propagation for a while after a large peak has been applied. These effects are allowed for in the more complex models.

## 6.8 Example: A Simple Damage-Tolerant Design Calculation

A pressure vessel is fabricated in an alloy steel using welded construction and is inspected before delivery. The limit of crack detection is 5mm, that is a 5mm defect can be missed. The vessel is pressurised to give a calculated hoop stress of 98MPa. Given that the fracture toughness of the material is  $100\text{MPam}^{1/2}$ , and the postulated defect is characterised as a surface crack in a semi-infinite body, what is the risk of fracture in service, what is the risk of fracture in a pre-service proof test at 50% over pressure and what would be the defect size required for fracture?

The  $K$  solution for a surface crack in a semi-infinite body is:

$$K = 1.12 * s * (p a)^{0.5} \text{ MPam}^{1/2}$$

which involves the three points of the triangle,  $K$ , crack size ( $a$ ) and stress ( $s$ ). The stress is given and the crack size is assumed to be the biggest defect that it is possible to miss, 5mm

At the operating pressure, the stress intensity on the notional 5mm deep defect is:

$$K = 1.12 * 98 * (3.1416 * 5 / 1000)^{0.5} = 13.76 \text{ MPam}^{1/2}$$

The proof test at 50% over pressure gives  $20.63 \text{ MPam}^{1/2}$ ; both values are much less than the  $K_{Ic}$  of  $100 \text{ MPam}^{1/2}$  so there is no risk of immediate fracture on

introduction to service or in the proof test if no defects greater than 5mm deep have been found.

Maintaining the stress as 98MPa, and finding the value for  $a$  which makes  $K$  equal to  $K_{Ic}$ , allows us to solve the triangle again to calculate the crack size required for fracture:

$$a = 1000 * ( 100 / \{ 1.12 * 98 \} )^2 / 3.1416 = 264mm$$

Indeed, the triangle can be worked again to prove that if the vessel survives the proof test, then it does not contain any defects greater than 117mm

## 6.9 Example: Solving the Box for Fatigue Crack Growth Assessments

This example follows on from the example above and concerns the same pressure vessel. Given the additional information that the threshold for crack growth is  $7MPam^{1/2}$  and that the growth of cracks by fatigue in this steel can be approximated by the simple Paris Law:

$$da/dN = 10^{-11} * ( DK )^3 \text{ metres/cycle}$$

it is required to assess the durability, that is the tolerance to fatigue crack growth, for start up/shut down cycling 20 times a day. Particular questions to solve once a relationship between the four corners of the box is established are:

- what is the life in years?
- what is the crack size after the design life of 25 years?
- if the final crack size is governed by plastic collapse to be 100 mm, what is the allowable initial defect size for a 25 year life?
- what is the chance that the vessel will survive a second proof test, for re-lifing purposes, after 25 years?

Firstly, we must establish a mathematical relationship between the parameters at the four corners of the rectangle, and this is obtained by direct substitution of the  $K$  solution into the Paris Law:

$$da/dN = 10^{-11} * (1.12 * \Delta S * \pi^{0.5} )^3 * a^{3/2}$$

Rearranging and integrating (the reader may like to attempt this as an exercise) leads to:

$$N_f = 25 * 10^9 * [ 1 / a_i^{0.5} - 1 / a_f^{0.5} ] / \Delta S^3$$

which is the relationship required between stress range ( $\Delta S$ ), life ( $N_f$ ), initial crack size ( $a_i$ ) and final crack size ( $a_f$ ). Now we can answer the questions by manipulation of this relationship.

By substituting 98MPa for the stress, 5mm for the initial size and 264 mm for the final crack size, the predicted crack growth life to fracture is 331,277 cycles i.e. 45.4 years at 20 cycles per day. Note that the final crack size has very little influence, indeed a finite life of 52.6 years is given for an infinite  $a_f$ . Put another way, if the final crack size is unknown, a value of infinity can be used.

By substituting 25 years (182,500 cycles) for life, we can find the "final" crack size at the end of the design life to be 18.15mm.

By substituting 25 years (182,500 cycles) for life and 100mm for the final crack size governed by plastic collapse, we can derive the allowable value of  $a_i$  to be 10.24mm.

To assess the risk of fracture in a second proof test to check if the vessel can be relined for continued use after 25 years, we must return to the triangle. The stress intensity on the 18.15mm crack in the proof test would be  $39.3 \text{ MPam}^{1/2}$ , which is still less than the fracture toughness of  $100 \text{ MPam}^{1/2}$ . Thus the vessel could be relined for a further 20 years on the basis that there cannot be any defects bigger than 117mm if the vessel passes the second proof test. However, the possibility of the toughness degrading in service due to thermal effects should be checked by further testing of the 25 year old material.

Similarly, the possibility of stress corrosion cracking should also be considered since the  $K_{ISCC}$  for such a steel could easily be as low as  $40 \text{ MPam}^{1/2}$ .

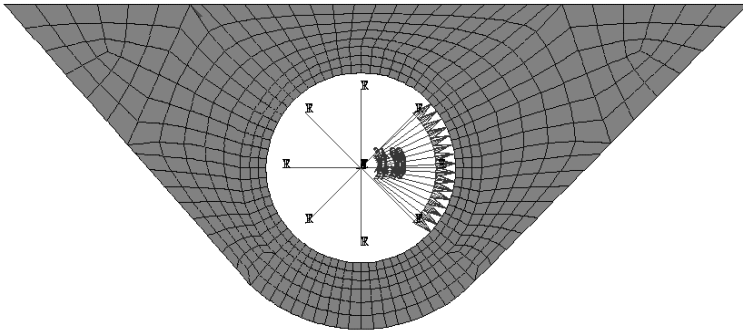
It should be recognised that the fatigue crack growth analyses above are only possible when the loading is constant amplitude and when both the  $K$  solution and the growth law are simple functions. For the more usual case of random loading, analysis must be on a cycle by cycle basis although a weighted average stress range (e.g. rms or root mean square of stress ranges) could be used as a constant amplitude equivalent. When the  $K$  solution is a complex function of crack size integration over a large number of small growth steps in each of which the  $Y$  function is assumed constant could be used. It should also be remembered that a semi-infinite body was assumed and so the  $Y=1.12$  used in this example is only valid for "short cracks" and increases with  $a/c$  ( $c$  being the thickness of the pressure vessel).

## 6.10 Aircraft Engine Mounting Lug FE Based Fatigue Analysis

In this section an aircraft engine-mounting lug is analyzed using the S-N approach. This is then followed by a Crack-Propagation analysis at the critical location identified in the S-N analysis. All the analysis was done using MSC.Fatigue.

### 6.10.1 S-N Analysis

The model that was analysed is shown in Figure 50. Two time histories of loading were specified. However, because of the nature of the load transferral from the centre pin to the mounting lug these two histories had to be separated into their positive and negative components. Figure 50 shows the loading distribution on the model for the x component in the positive direction. Figure 51 shows all four loading time histories.



*Figure 50. An aircraft engine lug model showing the loading distribution for the x component of load in the positive direction*

Figure 52 shows the static stress distribution caused by the y component of loading in the negative direction. Four such distributions are necessary, one corresponding to each loading time history shown in Figure 51.

The calculation method is relatively straightforward. Firstly the stress tensor history is calculated by linear superposition using the expression given in section 2.2.

An equivalent stress is calculated, in this case the largest principal stress. The stress history is then Rainflow counted and the damage calculated using the Palmgren-Miner rule with mean stress correction.

The material (S-N) curve used is shown in Figure 53. The process is repeated at every node and the resultant fatigue life distribution is shown in Figure 54. Note

that, since it is obvious that the fatigue life will be worst at a point on the surface, only surface elements have to be analysed by selecting these into an FE group.

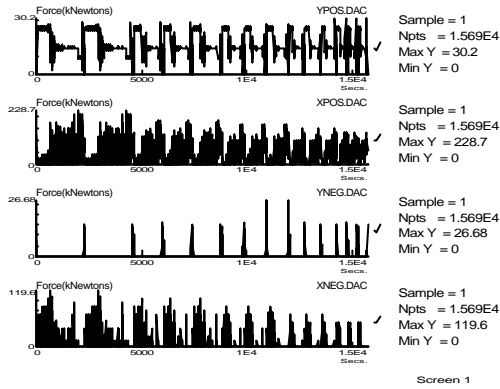


Figure 51. The four loading time histories applied in the x (+ve and -ve) and y (+ve and -ve) directions

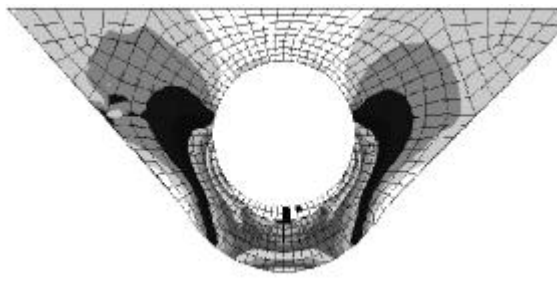


Figure 52. The stress distribution caused by one static load

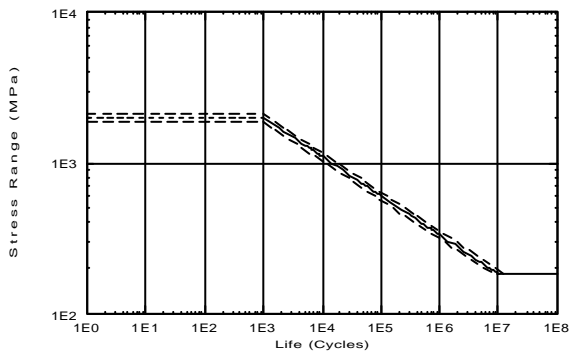


Figure 53. The S-N curve used (showing 5%, 50% and 95% failure lines)

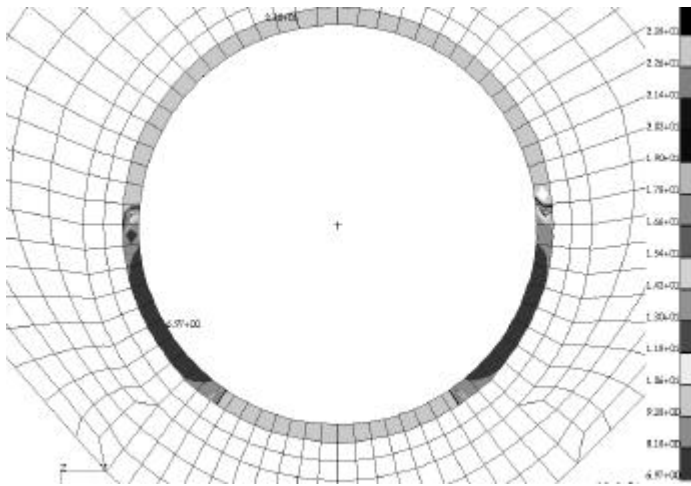


Figure 54. The S-N fatigue life predictions (log of life)

### 6.10.2 Crack Growth Prediction

Based on the results of the S-N analysis a critical location is identified. For this point an equivalent (largest principal stress) history is calculated, based on a nominal area around the critical node. This stress history is then time-cycle counted. A Compliance-Function for this particular geometric configuration is defined (as shown in Figure 55). This is one of many published Compliance Functions available in the software product used for this particular analysis (MSC.Fatigue). These were generally obtained from text books and other publications. Then, the crack growth is predicted using a cycle by cycle approach.

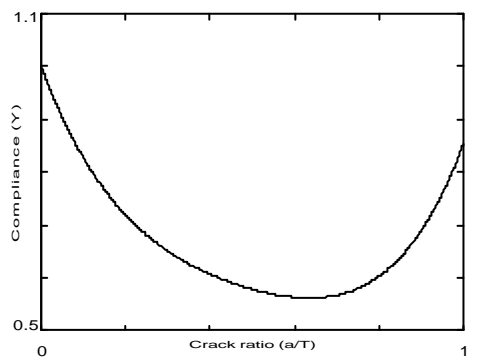


Figure 55. Compliance-Function for crack growing through lug

An effective stress intensity is used as shown in Figure 56. This allows history and other effects to be modelled. This figure shows the material curve for different mean stress levels (R values).

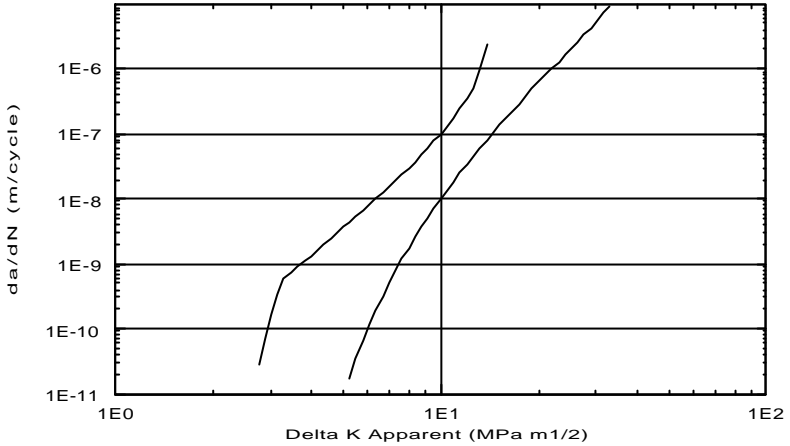


Figure 56. Delta K Apparent curves

The actual crack growth prediction for the critical location is shown in Figure 57.

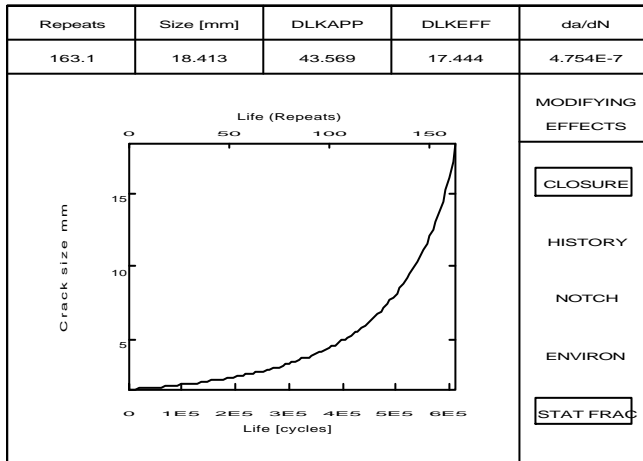


Figure 57. Crack growth history

## **6.11 FE Hints and Tips – Crack Growth**

### **6.11.1 The Use of Crack Tip Elements**

There is often some confusion about the use of crack tip elements for performing a crack growth analysis with FE. In general, fatigue calculations performed with FE results will combine linear elastic stress distributions calculated from an FE model, with a Compliance-Function, which has to be generated for the particular geometry being analysed. Hopefully this Compliance-Function already exists but if not the tools available for generating it include FE (in which case crack tip elements may be used), boundary element analysis and weight functions. In this regard the Compliance-Function is a geometry-based function that need only be created if it does not already exist. For a normal design, therefore, the FE model is used as a tool for generating stress distributions for an un-cracked state and these are then used for the fatigue design. The consequence of this “uncracked” assumption should always be assessed. (See also section 6.6).

### **6.11.2 Picking a Nominal Stress**

The stress to be used for any crack growth calculation should, for the reasons specified in the last section, be the nominal stress with no crack present. In fact, without any of the features that were present when the Compliance-Function was generated. In other words, this is a reference stress. Furthermore, the same reference stress position that is used as a parameter in the Compliance-Function curve must also be used in the FE stress analysis. (See also section 6.5).





# 7 Multi-axial Fatigue Analysis.

The Stress-Life, Strain-life and Fracture Mechanics methods described above can be regarded as well-established techniques that have been tried extensively in design situations. Details of the calculations are unlikely to change significantly and the software is stable. Other fatigue situations are not as well developed. In this case some of the calculations may be more speculative and the designer may have to become involved in choosing from a number of approaches. It is likely that a wider technical knowledge will be needed and new topics learned. One example of this is the situation when the stress state is multi-axial and this is covered briefly in this section. Note that in some parts of this section, and in line with normal convention, response stress states are sometimes referred to as “loadings”. This is somewhat confusing when used in an FE environment because loading generally means the input to an FE model.

## 7.1 Multi-axial Stress-Strain States

In a general three-dimensional state of stress there are nine components, being a normal stress and two resolved shearing stresses on each of three axes. In spite of this the Stress-Life, Strain-Life and Fracture Mechanics approaches all concentrate on a single stress, usually the normal stress on a fixed plane, but sometimes the shearing stress on a plane, also in a fixed direction. Mixed-mode fracture mechanics deals with the mechanism of crack growth from Stage I, through Stage II and on to the high strain shear modes that sometimes occur (Stage I and II crack growth, and the various modes of cracking, are described in sections 6.1 and 6.2).

In many real design situations more than one of the nine stresses are non-zero. In the simplest of these cases the stresses vary in simple proportion, so that the directions of principal planes at any point remain constant with time. In the most general case they do not vary in a proportional way, however, and at any point the directions of the principal stresses will vary during the cycle and are therefore a function of time. In this case many different planes are candidates for the start of failure.

Loadings (stress responses) can therefore be classified as:-

- (i) Uniaxial
- (ii) Multi-axial proportional
- (iii) Multi-axial non-proportional

The difficulty of making life estimations increases from (i) to (iii). Uniaxial stress response can be regarded as a controlled situation that has generally passed through the research stage into accepted design methodology. There are techniques for dealing with multi-axial proportional loading that have been applied successfully in design situations, although we are not confident about their accuracy and range of application. Non-proportional multi-axial loading is still the subject of active research.

As stated above, it is important to distinguish between multi-axial stresses and multi-axial loading. It is quite common to have more than one load application point, ie multi-axial loading, but at the same time have a uni-axial stress state because of the influence of component geometry.

### 7.1.1 Separate Tensile and Torsion Loading

In the most general situation, stress states are 3 dimensional. However, fatigue almost invariably starts at a surface, reducing the stress state to a two dimensional one. In this case the Mohr circle construction provides a good visualisation of the system. Two Mohr's circle representations of the stress-strain state prevailing under a static load are given in Figure 58 and Figure 59, for tension and torsion respectively.

For uni-axial tension,

$$\mathbf{s} = \mathbf{s}_1, \mathbf{s}_2 = \mathbf{s}_3 = \mathbf{0} \text{ and } \mathbf{e} = \mathbf{e}_1, \mathbf{e}_2 = \mathbf{e}_3 = -n\mathbf{e}_1.$$

For pure shear as in the case of a torsion test,

$$\mathbf{s} = \mathbf{s}_1 = -\mathbf{s}_3, \mathbf{s}_2 = \mathbf{0} \text{ and } \mathbf{e} = \mathbf{e}_1 = -\mathbf{e}_3, \mathbf{e}_2 = \mathbf{0}.$$

During cyclic loading, the diameter of the Mohr's circle varies depending on the magnitude of the cyclic stress or strain. Figure 60 illustrates this effect for a bar subjected to a constant amplitude uni-axial strain cycle.

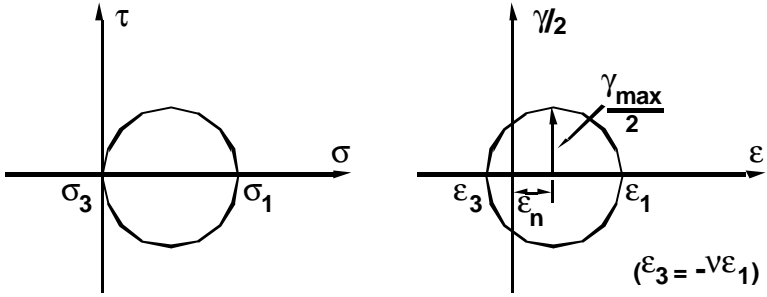


Figure 58. The stress-strain state in tension

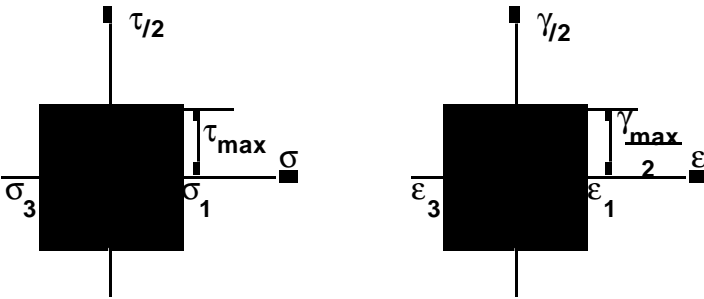


Figure 59. The stress-strain state in torsion

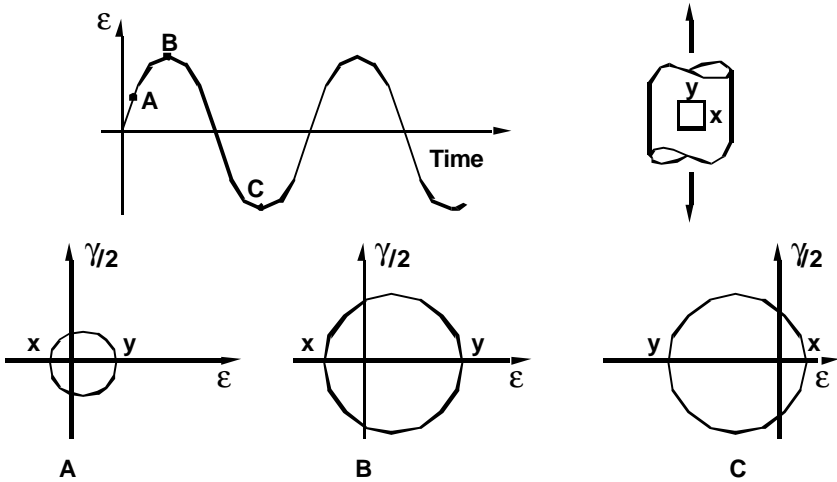


Figure 60. Mohr's Circle for a uni-axial strain cycle

### 7.1.2 Combined Tensile and Torsional Loading

It often happens in practice that separately applied load cases can result in *non-proportional* stress-strain states. Figure 61 can be used to illustrate the point by detailing the variation of axial and torsion strain with respect to time during the course of a complete fatigue cycle. Note that the loading path is no longer a straight line between normal and shear strain but rather a rectangular function. Note also that the direction of the principal strain is no longer fixed (i.e., the line  $x$ - $y$  subtends an angle  $\alpha$  with the strain axis).

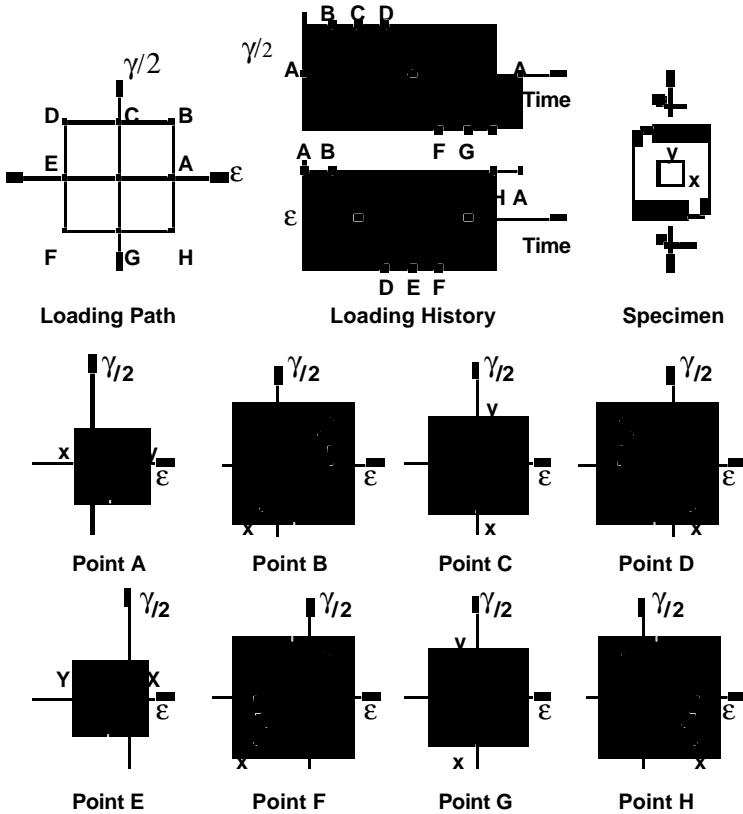


Figure 61. Non-proportional response sequences

### 7.2 Characterisation of Stress States

Whether or not a loading (stress response) is proportional and how severely it departs from the proportional case can be determined by observing the variation, with time, of stress ratios and principal stress directions. With most FE models the

stresses are resolved onto the element surface and so the third principal stress is zero. It should be noted that the 3<sup>d</sup> principal could be non-zero if an external pressure is being applied. However, this is not a common situation and then we can define the following two parameters.

$$\text{Elastic Biaxiality Ratio } a_e = \frac{S_2}{S_1}$$

$$\text{Principal Stress Angle } f_p$$

These parameters can then be tracked with time. The plots obtained for near proportional and non proportional stress-strain states are shown in Figure 62, Figure 63, Figure 64 and Figure 65.

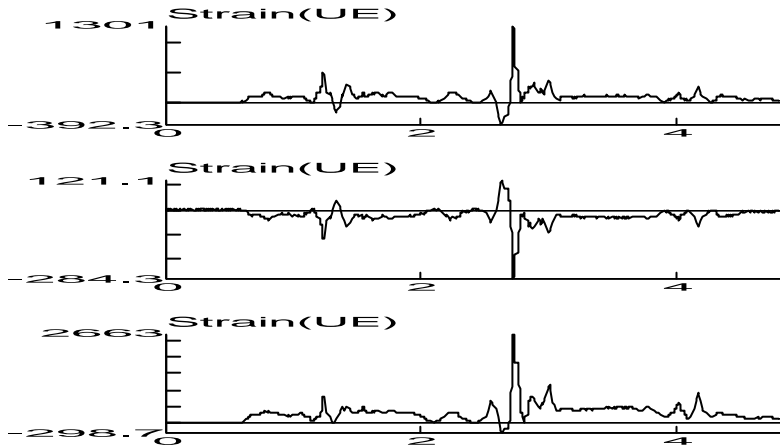
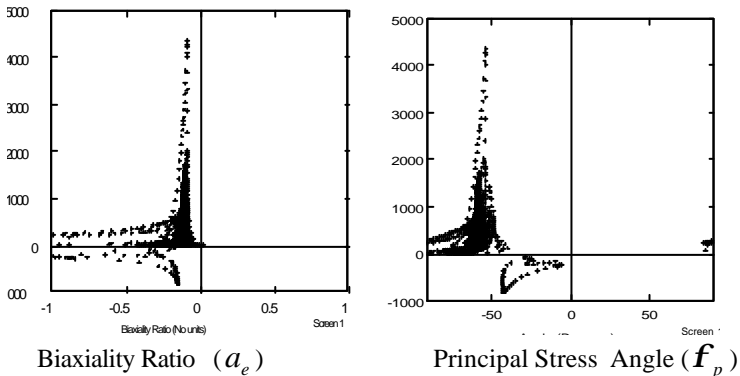


Figure 62. Near proportional response



Biaxiality Ratio ( $a_e$ )

Principal Stress Angle ( $f_p$ )

Figure 63. Near proportional Response

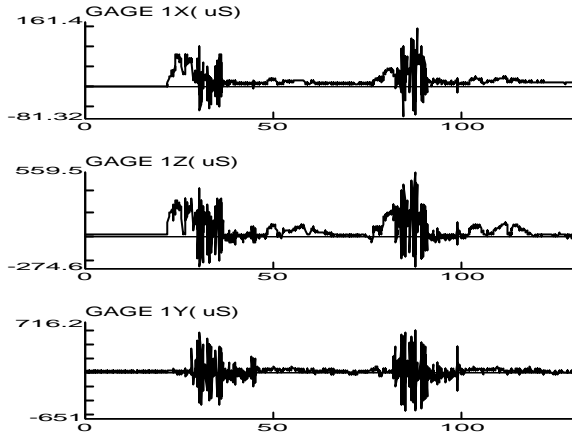


Figure 64. Non-proportional response

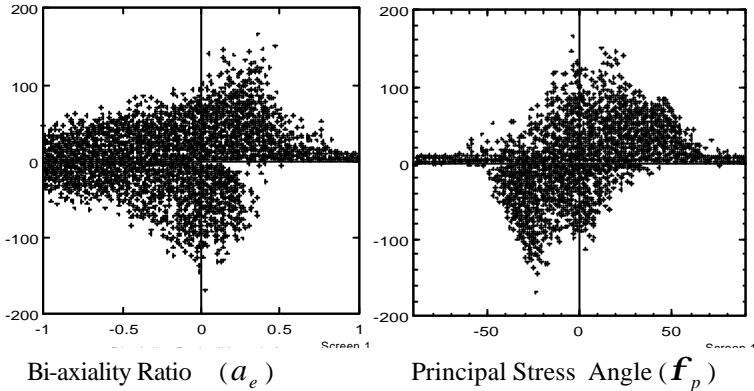


Figure 65. Non-proportional response

Using the values for the bi-axiality ratio and principal stress angle a determination can be made about the appropriate fatigue approach to use as follows,

Uni-axial       $f_p = \text{const}$        $a_e = 0$       uni-axial theories

Prop multi-axial       $f_p = \text{const}$        $-1 < a_e < 1 = \text{const}$       equiv stress-strain

Non-proportional       $f_p = \text{may vary}$        $a_e \text{ may vary}$       critical plane etc.

A description of equivalent stress-strain approaches and critical plane methods is given below.

### 7.3 Proportional Multi-axial Responses

Figure 66 shows that, even with proportional multi-axial response, there are then three distinct bands that the results can fall in to.

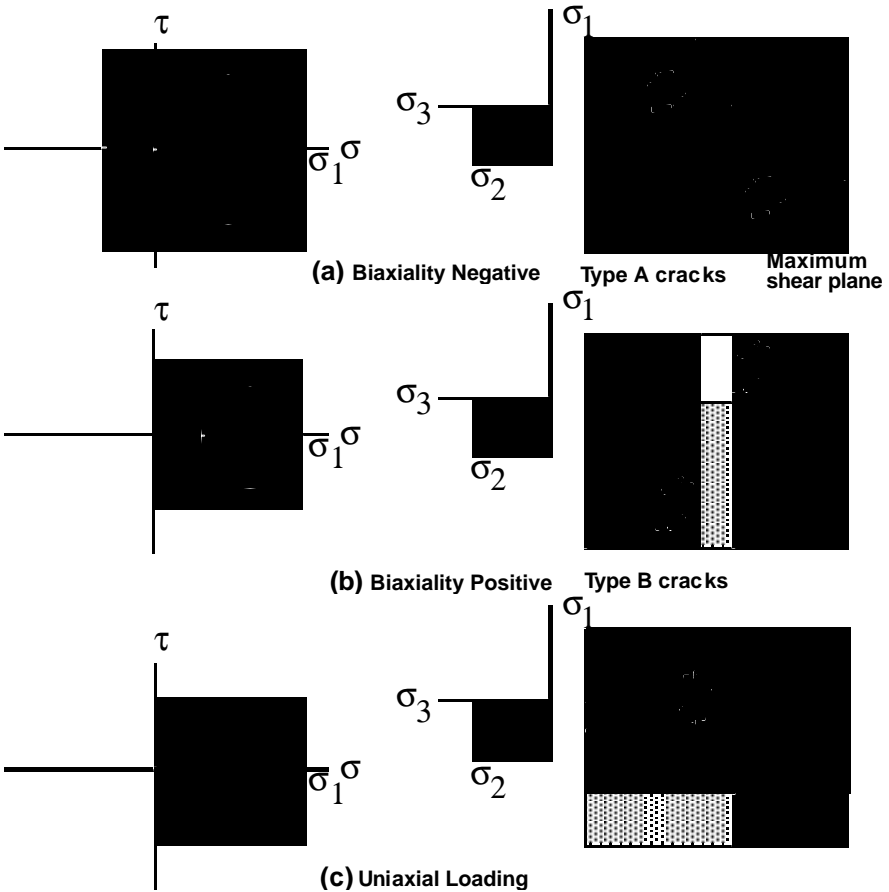


Figure 66. A check on the Bi-axiality ratio value

In Figure 66(a) the biaxiality ratio  $a_e$  lies somewhere between 0 and -1. A value of 0 represents the special case of uni-axial response, and -1 represents pure shear. In this range, the maximum shear strain occurs on a plane intersecting the free surface (shaded) at right angles, and cracks in the initiation phase will tend to be driven along the surface in mode II. These are Type A cracks.



In Figure 66(b),  $a_e$  lies between 0 (uni-axial loading) and 1 (equi-biaxial loading). The plane of maximum shear now intersects the surface at 45 degrees, and cracks growing in mode II will be driven through the thickness of the material. These Type B cracks are therefore more damaging for a given range of maximum shear strain.

In Figure 66(c) the special case of uni-axial response is depicted where the maximum shear plane is any plane at 45 degrees to the loading axis. Note that what we call **crack initiation** may actually include crack growth in shear and opening modes on more than one plane.

It is not always clear which type of crack behaviour dominates initiation. This is both material and stress-state dependent. The results of fatigue tests conducted under proportional loading at various bi-axiality ratios suggest that the best parameter to use for stresses and strains with  $0 < a_e < 1$  (i.e., between uni-axial and pure shear) is the absolute maximum principal strain.

For  $0 < a_e \leq 1$  (i.e., between uni-axial and equibiaxial) it is better, and more conservative, to use the signed Tresca parameter (section 7.4). This reflects that we are dealing with the more damaging type B cracks. The signed von Mises stress tends to give results that lie between the absolute maximum principal and the signed Tresca, except under uni-axial conditions, where all three should give the same result, and under equi-biaxial conditions where the signed von Mises and signed Tresca should give the same result.

There is a problem with responses where the value of  $a_e$  is around  $-1$ . Here the in-plane principal stresses are almost equal in magnitude but opposite in sign. The absolute maximum principal may flip between one and the other, and this may make it difficult to carry out a valid fatigue calculation. Under these circumstances, the angle-spread results will indicate that the angle is moving through 90 degrees or more. The designer may be able to get around this problem by using one of the strain components in conjunction with suitable fatigue data. These guidelines are summarised in the following table.

	<b>Abs Max Princ Strain</b>	<b>Signed von Mises</b>	<b>Signed Tresca</b>
<b><math>a_e = -1</math> (pure shear)</b>	<i>Possible problems with mobility of principals Use Critical Plane Option</i>		
<b><math>1 &lt; a_e &lt; 0</math></b>	<i>OK</i>	<i>Conservative</i>	<i>Very Conservative</i>
<b><math>a_e = 0</math> (uni-axial)</b>	<i>OK</i>	<i>OK</i>	<i>OK</i>
<b><math>0 &lt; a_e &lt; 1</math></b>	<i>Non-Conservative</i>	<i>Non-Conservative</i>	<i>OK</i>
<b><math>a_e = 1</math> (equibiaxial)</b>	<i>Non-Conservative</i>	<i>OK</i>	<i>OK</i>

To summarise the approach to assess for multi-axiality:

- (a). Calculate surface resolved stresses. These will be the principal stresses if no shear forces are acting on the surface. For shell elements this is normally not necessary in that the stresses are already surface resolved.
- (b). Run a global Strain-Life or Stress-Life fatigue analysis using the basic methods in sections 4 and 5.
- (c). Evaluate the mean biaxiality ratio, standard deviation of the mean biaxiality ratio, and the angle spread at the critical locations of the model as previously described.

If uni-axial conditions exist, no further steps are necessary.

If a proportional loading (stress) state exists in the critical locations, use equivalent stress-strain approaches.

If a non-proportional loading (stress) exists in the critical locations, then one of the more advance multi-axial techniques described next must be used, such as a critical plane approach.

#### **7.4 Equivalent Stress-Strain Approaches**

Traditionally, the approach to the design of components subjected to multi-axial loading is to make the following fundamental assumption:

Failure under a multi-axial loading is predicted to occur, according to the theory associated with a particular *modulus*, if and when the cyclically induced magnitude of that modulus, is sufficiently large that failure would occur in the uni-axial state for an identical magnitude of the same modulus.

The mechanical modulus referred to above, is a measurable quantity such as principal stress, principal shear stress or distortion energy.

This philosophy has lead to what is usually referred to as *equivalent stress-strain* approach. This is where an equivalent stress or strain modulus is calculated under multi-axial loading and then applied to uni-axial data.

These approaches are based on extensions to static yield theories. They assume that lifetimes for fatigue under multi-axial loading can be predicted by substituting combined stress or strain parameters in the uni-axial Stress-Life or Strain-Life equations (i.e., by calculating an equivalent uni-axial stress or strain for a given

multi-axial situation). The main stress and strain parameters used are the maximum principal, the maximum shear (related to the Tresca criterion), and the von Mises or octahedral. The big advantage of this kind of approach is that it enables the large amount of uni-axial fatigue test data available in the literature or in data banks to be applied to multi-axial situations.

The von Mises method has gained widest acceptance, but all these methods have drawbacks. In the Tresca criterion, the median principal stress does not affect the equivalent stress or strain, and neither of the von Mises or Tresca criteria varies with the application of a hydrostatic stress, contrary to experimental evidence.

The von Mises' prediction of yield in terms of the principal stresses is

$$\mathbf{s}_0 = 0.7071 [(\mathbf{s}_1 - \mathbf{s}_2)^2 + (\mathbf{s}_2 - \mathbf{s}_3)^2 + (\mathbf{s}_3 - \mathbf{s}_1)^2]^{0.5}$$

and yielding will occur when  $\mathbf{s}_0$  exceeds the monotonic yield stress. In terms of the (x,y,z) component stress, it may be written as:

$$\mathbf{s}_0 = 0.7071 [(\mathbf{s}_x - \mathbf{s}_y)^2 + (\mathbf{s}_y - \mathbf{s}_z)^2 + (\mathbf{s}_z - \mathbf{s}_x)^2 + 6(\mathbf{t}_{xy}^2 + \mathbf{t}_{yz}^2 + \mathbf{t}_{zx}^2)]$$

In order to perform a fatigue analysis the equivalent (uni-axial) strain parameter that is used is as follows.

$$\mathbf{e}_0 = \frac{\mathbf{s}_1}{|\mathbf{s}_1|(1+n)\sqrt{2}} [(\mathbf{e}_1 - \mathbf{e}_2)^2 + (\mathbf{e}_2 - \mathbf{e}_3)^2 + (\mathbf{e}_3 - \mathbf{e}_1)^2]^{0.5}$$

## 7.5 Dealing with Non-Proportional Responses

A problem with the equivalent stress-strain methods is that they do not take into account the fact that fatigue is essentially a directional process, with damage (i.e., cracking) taking place on particular planes. In addition, there are problems in applying any of the methods to situations of non-proportional loading (i.e., where the number of load inputs is greater than one, and these loads have a non-constant ratio or phase relationship).

This has led to much greater research emphasis being placed on understanding the underlying mechanisms of fatigue damage accumulation under multi-axial loading (stresses). Out of this work has arisen a somewhat different approach to fatigue life estimation based on predicting the extent of damage in specific directions and planes within the component. This methodology is usually referred to as the *critical plane* approach.

As the name suggests, critical plane approaches recognise that fatigue damage (cracking) is essentially directional and so consider the accumulation of damage on particular planes. This is in contrast to the equivalent stress-strain approaches, which may be summing damage that is occurring on different planes. A range of methods for calculating damage on a particular plane are used. These methods include:

- (a). Simply resolving the normal strain on the plane and applying what is essentially a uni-axial analysis.
- (b). As above, but allowing for the multi-axial nature of the stresses through a consideration of the out-of-plane hardening.
- (c). Variations on the Brown-Miller approach, which consider the shear strain amplitude in the plane and the normal stress or strain.
- (d). Typically, damage is calculated for all possible planes (at, say 10 degree intervals) and the worst or critical plane selected.

Fatigue cracks normally initiate on planes of maximum shear and grow initially in mode II (shear mode), changing to mode I (opening mode) where the maximum principal may be important. In brittle materials or where initiation occurs at inclusions, growth may be in mode I from the outset. The term *initiation* is usually used to refer to the time taken to develop an *engineering crack* and so this so-called initiation phase may include both stages of crack development. This definition is different from the materials scientists one which usually refers only to Stage I (Mode II). In practice, Stage I or Stage II may dominate lifetime. In uni-axial fatigue, this is not so much of a problem, as the controlling parameters in both cases are directly related to the uni-axial stress or strain.

In multi-axial fatigue the fundamental process is still the same. However, the fact that multiple planes of crack growth can occur does complicate the process and so it is very difficult to apply general rules and a model that produces good predictions for a given material and loading may no longer hold true when applied to a different situation. To add further confusion, there may be interaction between damage on different planes; cracks growing on a particular plane may actually impede the progress of those growing on a different plane and lead to an increase in fatigue life.

Most critical plane approaches use the Strain-Life damage method. However, there is one critical plane approach, the McDiarmid criterion, which uses a High Cycle Fatigue limit concept. Another fatigue limit approach, the Dang-Van Criterion, uses the maximum microscopic shear stress and the hydrostatic stress, as damage defining parameters.

## **7.6 FE Hints and Tips – Multi Axial Fatigue**

### **7.6.1 3D, 2D and 1D Models**

Most FE models that appear to be 3D often become 2D or even 1D when a fatigue calculation is performed. Consider first of all that fatigue cracks nearly always initiate on the surface of the material. For this reason it is usually acceptable to ‘skin’ a 3D model with 2D elements and then only use these elements in any subsequent fatigue analysis. Now consider that, in many cases, fatigue critical locations occur in regions of high stress and very often these regions are constrained in such a way that the stress states are also constrained into 1 direction. So, in this case, the 3D stress state actually becomes a 1D stress state. This is often a useful simplifying factor for designs but it also helps to explain why multi-axial fatigue calculations should be considered as a last resort.

### **7.6.2 Multi Modal Response to Single Input Loading**

Unfortunately, it is not possible to conclude that when an FE model has only one load input the stress response will always be uniaxial because more than one mode may be excited. Consider the simple case of a beam with a lateral load applied that is slightly offset from the beam central axis. If this loading has a frequency that is near to both a bend mode and a twist mode both modes will tend to be excited.

### **7.6.3 Summary**

Multiaxial fatigue life estimation techniques involve complicated procedures. The contents of this section have provided a review of the basic approaches available. It should be expected, that in the vast majority of cases, normal designers will not have to revert to such complexities and will be able to use the standard uni-axial techniques described earlier. Furthermore, practical applications of these more advanced approaches have shown that although they can influence the predicted fatigue life magnitude they often do not uncover (or overlook) potential hot spots identified by the simpler and quicker methods. So even if a multi-axial situation does exist, uniaxial predictions can still be utilised for relative analyses or robustness studies. However, it should be remembered that the basis of such a uniaxial approach would need to be verified by checking the stress conditions in critical locations. Section 7.2 describes two such checks for the occurrence of secondary principal stresses and for stress tensor rotation. This section should be referred to for more information.

# 8 Vibration Fatigue Analysis.

This chapter provides a state of the art review of vibration fatigue technology. The techniques described are of particular relevance where the input loading is exciting a resonance response in the system or the input loadings are too long to be stored as a time series and need to be stored in the form of a Power Spectral Density (PSD) function, otherwise good old linear elastic static analyses (with or without inertia relief loading), can be used.

It is necessary to clarify the term *vibration fatigue* as the estimation of fatigue life when the input loading, or the stress history obtained from the structure, or component, is random in nature and therefore best specified using statistical information about the process. This is usually in the form of a PSD function. The same approach can also be described using the terms *spectral fatigue analysis*, or *frequency based fatigue* techniques.

## 8.1 Alternative Descriptions of Engineering Processes

Most designers, if asked to specify a random loading input, or response output, for a structural system would specify the random time history shown in Figure 67. This process can be described as *random* and in the *time domain*.

The process is described as random because, strictly speaking, it can only be determined statistically. A second sample taken for the same process would obviously have different values to the first.

There are several alternative ways of specifying the same random process. Fourier analysis allows any random loading history of finite length to be represented using a set of sine wave functions, each having a unique set of values for amplitude, frequency and phase. Such a representation is called *deterministic* (Figure 68) because the individual sine waves can be determined precisely at any given point in time.

It is still time based and so is therefore specified in the time domain. As an extension of Fourier analysis, Fourier transforms allow any process to be represented using a spectral formulation such as a *Power Spectral Density PSD* function. Such a process is described as a function of frequency and is therefore said to be in the *frequency domain* (see Figure 67). It is still a random specification of the function.

For the vast majority of engineering problems, if you have one form of the above three loading specifications you can quite easily get to one of the two alternative specifications. These transformations rely on the assumption that the process is stationary, random and Gaussian (see later). Fortunately, most engineering processes conform reasonably well to these assumptions.

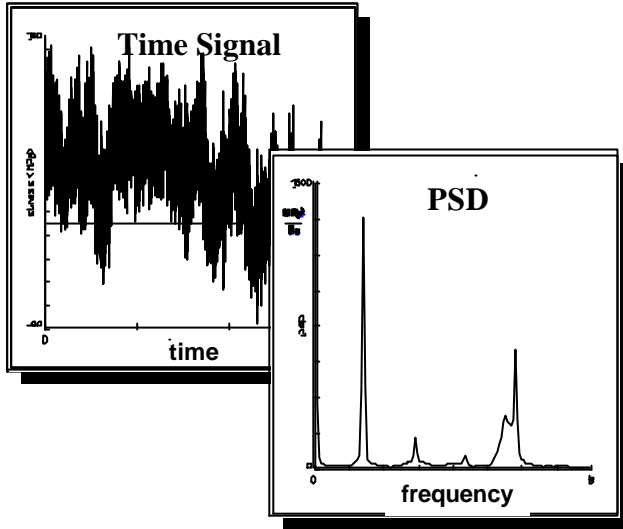


Figure 67. Random processes

Coupled with each of these three specifications for the loading or response are three alternative analysis types. The key question for a designer is therefore which type of structural analysis to use, and subsequently which type of fatigue analysis approach to use. For instance, the environmental conditions experienced by aircraft structures can be represented using either a discrete (deterministic) gust approach, or a continuous gust spectrum of atmospheric turbulence.

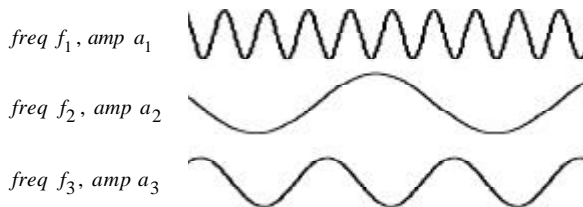


Figure 68. Deterministic processes

However, wherever dynamic response is present it is usually desirable, where possible, to do the structural analysis in the frequency domain using PSDs and transfer functions.

### 8.1.1 What is the Frequency Domain?

Structural analysis can be carried out in either the time or frequency domains as shown in Figure 69. In the time domain the input takes the form of a time history of load (in this case wind speed). The structural response can be derived using a finite element representation coupled with a transient (convolution) solution approach. The output from this model is also expressed as a time history, in this case the stress at some particular location in the structure.

In the frequency domain the input is given in the form of a PSD of wind speed and the structure is modelled by a linear transfer function relating input wind speed to the output stress at a particular location in the structure. The output from the model is expressed as a PSD; in this case it is the PSD of stress.

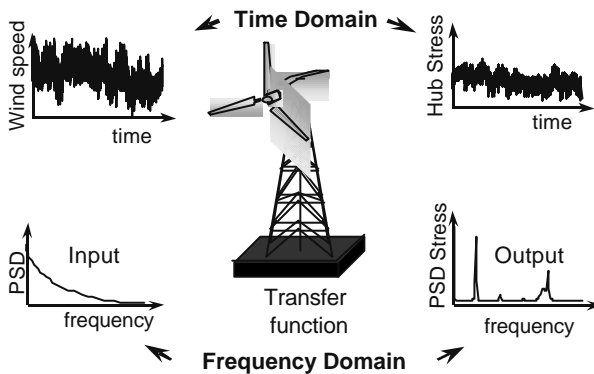


Figure 69. What is the frequency domain?

Most of the computational time is spent in solving the structural model. In the time domain, the structural model is solved for each time history of input; hence 20 load cases would take 20 times as long to calculate as 1. In the frequency domain the linear transfer function is only calculated once, hence 20 load cases takes little more time to analyse than 1. Obviously, if we are calculating a linear structural model then the structure must behave linearly. Fortunately in most engineering situations this is a reasonable assumption.



### 8.1.2 What is a Power Spectral Density (PSD)?

PSDs are obtained by taking the modulus squared of the Fast Fourier Transform (FFT). The FFT outputs a complex number given with respect to frequency but in a PSD only the amplitude of each sine wave is retained (see Figure 70). All phase information is discarded.

Any periodic function can be expressed as the summation of a number of sinusoidal waves of varying frequency, amplitude and phase. In most engineering situations it is only the amplitude of the various sine waves that is of interest. In fact, in many cases we find that the initial phase angle is totally random, and so it is unnecessary to show it. For this reason the PSD function alone is usually used. One very useful characteristic can be calculated directly from the PSD. The so-called *root mean square (rms)* value is defined as the square root of the area under the PSD curve.

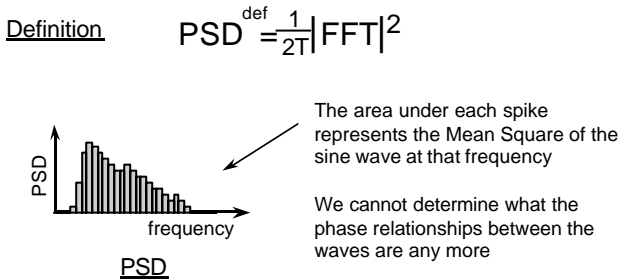


Figure 70. What is a PSD?

## 8.2 Characterization of Engineering Processes Using Statistical Measures

### 8.2.1 Time Histories & PSDs

Engineering processes can fall into a number of types and Figure 71 is useful as a means of characterising these different types of processes.

Figure 71(a), a *sinusoidal time history* appears as a single spike on the PSD plot. The spike is centred at the frequency of the sine wave and the area of the spike represents the mean square amplitude of the wave. In theory this spike should be infinitely tall and infinity narrow for a pure sine wave. However, because any sine wave used is, by definition, finite in length, the spike always has finite height and finite width. Remember, with PSD plots it is the area under the graph that is of interest and not the height of the graph.

Figure 71(b), a **narrow band process** is shown which is built up of sine waves covering only a narrow range of frequencies. A narrow band process is typically recognised in a time history by amplitude modulation, often referred to as a ‘beat’ envelope.

Figure 71(c), a **broad band processes** is shown which is made up of sine waves over a broad range of frequencies. These are shown in the PSD plot as either a number of separate response peaks (as illustrated) or one wide peak covering many frequencies. This type of process is usually more difficult to identify from the time history but is typically characterised by positive valleys (troughs in the signal above the mean level) and negative peaks.

Figure 71(d), a **white noise process** is shown. This is a special time history, which is built up of sine waves over the whole frequency range.

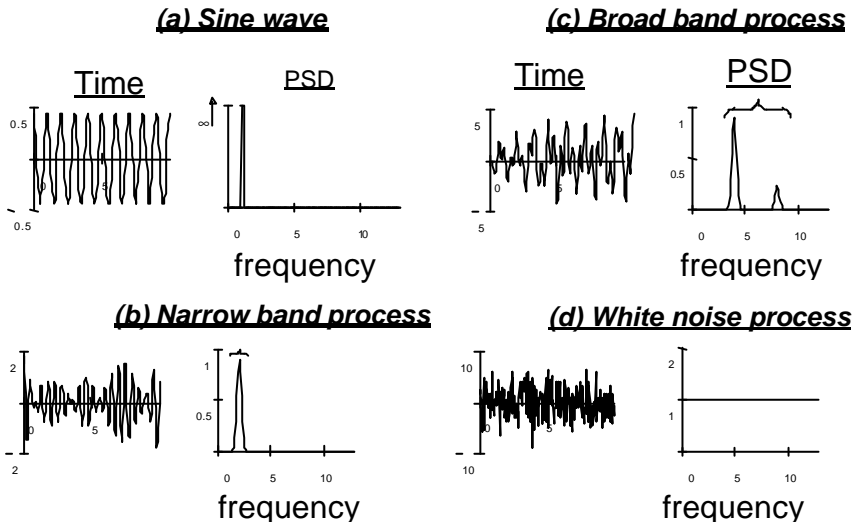


Figure 71. Equivalent time histories and PSDs

### 8.2.2 Expected Zeros, Peaks and Irregularity Factor

Random stress or strain time histories can only properly be described using statistical parameters. This is because any sample time history can only be regarded as one sample from an infinite number of possible samples that could occur for the random process. Each time sample will be different. However, as long as the samples are reasonably long then the statistics of each sample should be constant.

Two of the most important statistical parameters are the number of so-called **zero crossings** and **number of peaks** in the signal. Figure 72 shows a one second piece cut out from a typical wide band signal.

$E[0]$  represents the number of (upward) zero crossings, or mean level crossings for a signal with a non-zero mean.  $E[P]$  represents the number of peaks in the same sample. These are both specified for a typical 1 second sample. The irregularity factor is defined as the number of upward zero crossings divided by the number of peaks.

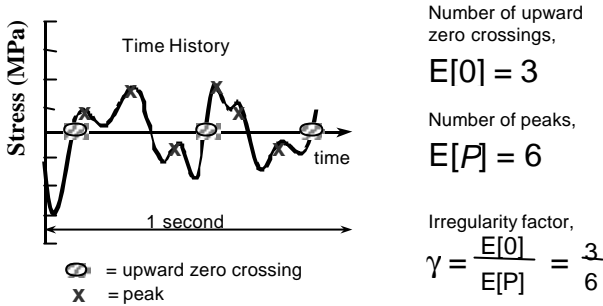


Figure 72. Zero and peak crossing rates

In this particular case the number of zeros is 3, and the number of peaks is 6, so the irregularity factor is equal to 0.5. This number can theoretically only fall in the range 0 to 1. For a value of 1 the process must be narrow band as shown in Figure 71(b). As the divergence from narrow band increases then the value for the irregularity factor tends towards 0.

### 8.2.3 Moments From a PSD

Since this section is concerned with structural systems analysed in the frequency domain a method is required for extracting the **probability density function pdf** of Rainflow ranges directly from the PSD of stress. The characteristics of the PSD that are used to obtain this information are the ***nth moments*** of the PSD function (Figure 73). In fact these moments provide all the information required to calculate fatigue damage. The relevant spectral moments are easily computed from a one sided PSD  $G(f)$  in units of Hertz using the following expression.

$$m_n = \int_0^{\infty} f^n \cdot G(f) df = \sum f_k^n \cdot G_k(f) \cdot df$$

The  $n^{\text{th}}$  moment of area of the PSD ( $m_n$ ) is calculated by dividing the curve into small strips as shown. The  $n^{\text{th}}$  moment of area of the strip is given by the area of the strip multiplied by the frequency raised to the power  $n$ . The  $n^{\text{th}}$  moment of area of the PSD is then found by summing the moments of all the strips.

In theory, all possible moments are required to fully characterise the original process. However, in practice, we find that  $m_0$ ,  $m_1$ ,  $m_2$  and  $m_4$  are sufficient to compute all of the information required for the subsequent fatigue analysis. This is quite amazing when one thinks about what can be obtained using just these 4 moments.

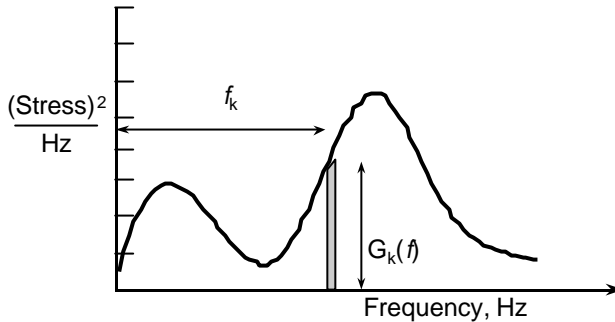


Figure 73. Calculating PSD moments

It is useful to note that the *root mean square* (rms) value is equal to  $\sqrt{m_0}$

#### 8.2.4 Expected Zeros, Peaks and Irregularity Factor From a PSD.

The first serious effort at providing a solution for estimating fatigue damage from PSDs was undertaken by SO Rice in 1954. Rice developed the very important relationships for the number of upward mean crossings per second ( $E[0]$ ) and peaks per second ( $E[P]$ ) in a random signal expressed solely in terms of their spectral moments  $m_n$ .

$$E[0] = \sqrt{\frac{m_2}{m_0}} \quad \text{and} \quad E[P] = \sqrt{\frac{m_4}{m_2}} \quad \text{Therefore} \quad \mathbf{g} = \frac{E[0]}{E[P]} = \sqrt{\frac{m_2^2}{m_0 m_4}}$$

#### 8.2.5 What is The Transfer Function?

If a sinusoidal force is applied to a linear structure then the structure will respond with a sinusoidal displacement at the same frequency. For a linear structure we can also expect that an increase in the amplitude of the forcing function will cause a

proportional increase in the structural displacement. This leads to the concept of a linear *transfer function*.

The transfer function is defined as the response per unit input at each frequency of interest as shown Figure 74. We can therefore use the transfer function to predict the amplitude displacement of the structure by multiplying the amplitude of the load  $F$  by the transfer function  $T$  for a particular frequency of applied load.

The transfer function can be calculated using a number of methods. These include computer generated models (FE analysis) and using data acquired from tests. An intuitive way to calculate it would be to apply a series of sine waves to a test rig, or an FEA model, and then find the amplitude of the structural response for each frequency. The structural response does not have to be in the form of displacement. It could be in the form of strain or stress provided that the relationships are linear. Similarly, the input may be wind speed, acceleration, wave height, etc. and need not be force, but again the relationships must be linear.

To get the transfer function into the correct units for a PSD analysis the response parameter (per unit input loading) has to be squared. This is because the units of PSDs are units of interest squared, per hertz. If one takes the example of an offshore platform the input loading is typically expressed as a sea state spectrum. The process that this PSD defines is the sea surface elevation profile. In the time domain this is the sea surface elevation variation with time. The units of the input PSD, transfer function and response PSD are therefore given as:

$$\text{Input PSD} \times \text{transfer function} = \text{response PSD}$$

$$\frac{m^2}{Hz} \quad \times \quad \left[ \frac{MPa}{m} \right]^2 = \frac{MPa^2}{Hz}$$

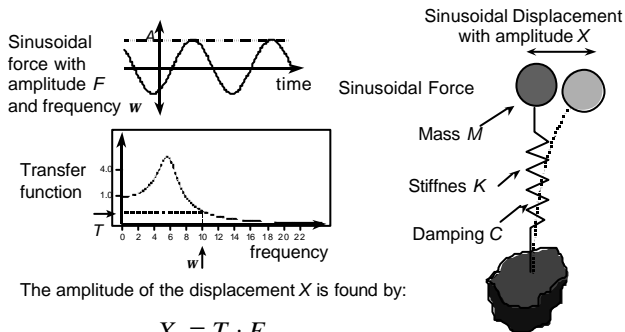


Figure 74. The concept of a linear transfer function

### 8.3 What is a Modal Transient (Superposition) Fatigue Analysis?

For the fatigue analysis of dynamic systems there are generally three main capabilities. Option 1 is a “dynamic transient” analysis which requires input time histories to be applied to the FEA model point by point and so can be very computationally intensive as well as requiring large file storage. If two load inputs (at the same time) are needed then these are applied simultaneously to the FEA model. If more than one so-called duty cycle is required (ie, load history one for 2 hours, load history 2 for 6 hours, etc) then each has to be applied separately therefore increasing the computation time.

The second option is a vibration fatigue analysis, which requires input PSDs, and cross PSDs, and the structural transfer functions computed by an FE model. Once these transfer functions are available any number of duty cycles can be accommodated with very little computational effort. This approach requires smaller file storage and is computationally more efficient.

Option 3, which is a hybrid version of the dynamic transient approach, uses so-called modal contribution factors from each input time history. These are then multiplied by the modal influence coefficients (mode shapes of the structure) to get time history responses for each point on the structure. Furthermore, for the knowledgeable user, modes can sometimes be removed to speed up the analysis. This is a more computationally efficient version of the transient approach and it can significantly reduce storage space requirements. However for each mode present in the model there has to be a subset of the original time history to represent the modal contribution factors. So, for example, if 20 modes are present then 20 subsets, each the same size as the original file, have to be created. Input data file storage can therefore sometimes be an issue to consider. However, by removing some modes the modal transient approach may be quicker than a conventional transient analysis. Furthermore, it can, unlike the frequency-based approach, retain the exact original time histories of load in the analysis. For this reason it is becoming a popular approach for the analysis of full body simulation problems.

In summary, therefore, a modal transient method is basically a modified transient analysis. But if modes are removed it can be much more computationally efficient. For the less sophisticated analyst the concept of mode removal will add an additional technical complication and so non-advanced users should proceed with care.

## 8.4 Fatigue Life Estimation From PSDs

Before introducing the concepts needed to estimate fatigue damage in the frequency domain it is useful to set out a parallel approach in the time domain. The approach highlighted is that of a traditional Stress-Life approach (see Figure 75).

### 8.4.1 Time Domain Stress-Life Fatigue Life Estimation

The starting point for any fatigue analysis is the response of the structure or component, which is usually expressed as a stress or strain time history. If the response time history is made up of constant amplitude stress or strain cycles then the fatigue design can be accomplished by referring to a typical S-N diagram. However, because real signals rarely conform to this ideal constant amplitude situation, an empirical approach is used for calculating the damage caused by stress signals of variable amplitude.

Despite its limitations, the Palmgren-Miner rule is generally used for this purpose. This linear relationship assumes that the damage caused by parts of a stress signal with a particular range can be calculated and accumulated to the total damage separately from that caused by other ranges (see section 4.6). A ratio is calculated for each stress range, equal to the number of actual cycles at a particular stress range,  $n$ , divided by the allowable number of cycles to failure at that stress,  $N$ , (obtained from the S-N curve). Failure is assumed to occur when the sum of these ratios, for all stress ranges, equals 1.0.

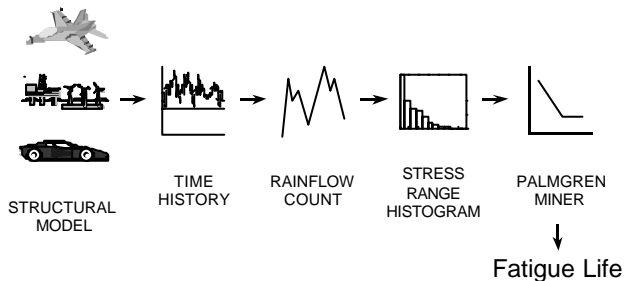


Figure 75. A standard S-N fatigue analysis

If the response time history is irregular with time, as shown in the Figure 75, then **Rainflow** cycle counting is widely used to decompose the irregular time history into equivalent sets of block loading. The numbers of cycles in each block are usually recorded in a stress range histogram. This can then be used in the Palmgren Miner calculation. An example of the way Rainflow ranges are extracted from a time signal is given in section 4.7.2.

### 8.4.2 S-N Relationship

A traditional SN curve as shown in Figure 76 is used to model the material properties of the components being analysed. This simply shows that, under constant amplitude cyclic loading, a linear relationship exists between cycles to failure  $N$  and applied stress range  $S$  when plotted on log-log paper. There are two alternative ways of defining this relationship, as given below ( $b$  and  $k$ , or  $m$  and  $SRII$ , are material properties).

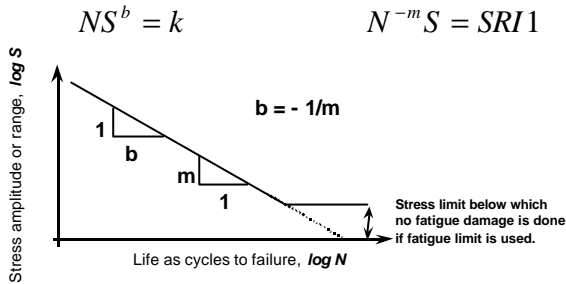


Figure 76. A typical S-N curve

### 8.4.3 Estimating Fatigue Life From a Stress pdf

*Probability density functions* (pdfs) are obtained by normalising any histogram so that the area under a continuous curve representing the height of each bin is equal to 1.0. Once the stress range histogram has been converted into a stress range pdf then there is an elegant and efficient equation to describe the expected fatigue damage caused by this loading history.

$$E[D] = E[P] \frac{T}{k} \int_0^{\infty} S^b p(S) dS$$

In order to compute fatigue damage over the lifetime of the structure in seconds ( $T$ ) the form of material (S-N) data must also be defined using the parameters  $k$  and  $b$  (or  $m$  and  $SRII$ ). In addition, the total number of cycles in time  $T$  must be determined from the number of peaks per second  $E[P]$ . If the damage caused in time  $T$  is greater than 1.0 then the structure is assumed to have failed. Or alternatively the fatigue life can be obtained by setting  $E[D] = 1.0$  and then finding the fatigue life  $T$  in seconds from the fatigue damage equation given above.

### 8.4.4 The Frequency Domain Model

Figure 75 highlighted the overall process for fatigue life estimation in the time domain. The parallel approach in the frequency domain is shown in Figure 77. If



we assume that the structural model shown is now an FEA model, this model would be identical for both the time domain and frequency domain approaches.

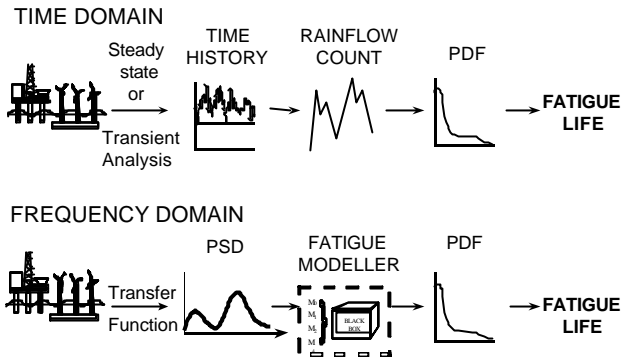


Figure 77. Time versus frequency domain calculations

In order to get structural response in the time domain a transient structural analysis would be required, before the fatigue analysis. In the frequency domain a transfer function would first be computed for the structural model. This is completely independent of the input loading and is a fundamental characteristic of the system, or model. The PSD response, caused by any PSD of input loading, is then obtained by multiplying the transfer function by the input loading PSD. Further response PSDs caused by additional PSDs of input loading can then be calculated with a trivial amount of computing time. Once the response PSD has been computed the remaining task is to estimate the fatigue damage using one of a number of fatigue models.

#### 8.4.5 Narrow Band Solution

Bendat presented the theoretical basis for the first of these of these frequency domain fatigue models, the so-called **Narrow Band** solution. This expression was defined solely in terms of the spectral moments up to  $m_4$ . However, the fact that this solution was suitable only for a specific class of response conditions was an unhelpful limitation for the practical engineer. The narrow band formula is given below.

$$\begin{aligned}
 E[D] &= \sum_i \frac{n_i}{N(S_i)} = \frac{S_t}{k} \int S^b \cdot p(S) \, dS \\
 &= \frac{E[P] \cdot T}{k} \int S^b \cdot \left[ \frac{S}{4m_0} e^{\frac{-S^2}{8m_0}} \right] \, dS
 \end{aligned}$$

This was the first frequency domain method for predicting fatigue damage from PSDs and it assumes that the pdf of peaks is equal to the pdf of stress amplitudes. The narrow band solution was then obtained by substituting the Rayleigh pdf of peaks with the pdf of stress ranges. The full equation is obtained by noting that  $S_T$  (the total number of cycles in time T) is equal to  $E[P].T$ , where T is the life of the structure in seconds (see Figure 78).

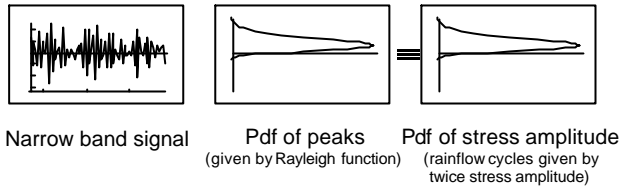


Figure 78. The basis of the narrow band solution

Figure 79 explains why the narrow band solution is so conservative for wide band cases? Two time histories are shown. The narrow band history (a) is made up by summing two independent sine waves at relatively close frequencies. The wide band history (b) uses two sine waves with relatively widely spaced frequencies.

Narrow banded time histories are characterised by frequency modulation known as a beat effect. Wide band processes are characterised by the presence of positive troughs and negative peaks and these are clearly seen in Figure 79 as a sinusoidal ripple superimposed on a larger, dominant sine wave.

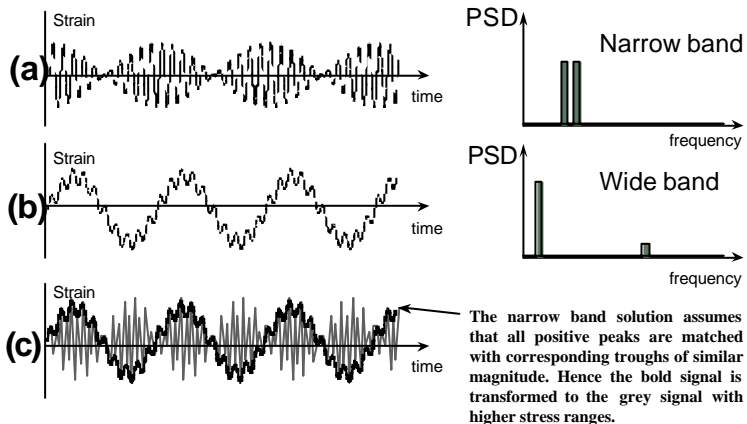


Figure 79. Why the narrow band solution is conservative

The problem with the narrow band solution is that positive troughs and negative peaks are ignored, and all positive peaks are matched with corresponding troughs

of similar magnitude regardless of whether they actually form stress cycles. To illustrate this, take every peak (and trough) and make a cycle with it by joining it to an imaginary trough (peak) at an equal distance the other side of the mean level. This is shown in Figure 79(c). It is easy to see that the resultant stress signal contains far more high stress range cycles than were present in the original signal. This is the reason why the narrow band solution is so conservative.

#### 8.4.6 Empirical Correction Factors (Tunna, Wirsching, Hancock, Chaudhury and Dover)

Many expressions have been proposed to correct this conservatism. Most were developed with reference to offshore platform design where interest in the techniques has existed for many years. In general, they were produced by generating sample time histories from PSDs using Inverse Fourier Transform techniques. From these a conventional Rainflow cycle count was then obtained. The solutions of Wirsching et al, Chaudhury and Dover, Tunna and Hancock were all derived using this approach. They are all expressed in terms of the spectral moments up to  $m_4$ .

The approach of Steinberg leads to a very simple solution based on the assumption that no stress cycles occur with ranges greater than 6 rms values. The distribution of stress ranges is then arbitrarily specified to follow a Gaussian distribution. This defines the stress range cycles to occur with a particular probability.

A paper by Bishop (1999) has full details of the above techniques.

#### 8.4.7 Dirlik's Empirical Solution for Rainflow Ranges

Dirlik has produced an empirical closed form expression for the pdf of Rainflow ranges, which was obtained using extensive computer simulations to model the signals using the Monte Carlo technique. Dirlik's solution is given below.

$$p(S) = \frac{\frac{D_1}{Q} e^{-\frac{Z}{Q}} + \frac{D_2 Z}{R^2} e^{-\frac{Z^2}{2R^2}} + D_3 Z e^{-\frac{Z^2}{2}}}{2(m_0)^{1/2}}$$

where  $x_m$ ,  $D_1$ ,  $D_2$ ,  $D_3$ ,  $Q$  and  $R$  are all functions of  $m_0$ ,  $m_1$ ,  $m_2$ , and  $m_4$ .

$Z$  is a normalised variable equal to  $\frac{S}{2(m_0)^{1/2}}$ .

### 8.4.8 Bishop's Theoretical Solution for Rainflow Ranges

Dirlik's empirical formula for the pdf of Rainflow ranges has been shown to be far superior, in terms of accuracy, than the previously available correction factors. However, the need for certification of the technique before its use meant that theoretical verification was required. This was achieved by Bishop, when a theoretical solution for predicting Rainflow ranges from the moments of the PSD was produced.

### 8.4.9 Clipping Ratio as a Function of rms

Most of the above techniques require an integration cut-off to be set in terms of the numbers of rms values along the stress range axis. It is normal to set this to 3 rms (for amplitude) or 6 rms (for range). However, it is easy to show, using measured signals of sufficient length, that up to 4.5 rms (on amplitude) is needed to avoid omitting fatigue-damaging cycles.

### 8.5 A Simple Vibration Fatigue Hand Calculation

In order to illustrate the mechanism of vibration fatigue calculations it is worth performing some simple hand calculations on the two-peaked PSD shown in Figure 80.

Approximate hand calculations have been performed in both the time and frequency domains. A computer-based calculation (using MSC.Fatigue) has also been performed as a comparison. This example is very useful for demonstrating that although the concepts underlying vibration fatigue tools may be complex it is still possible to use simple hand calculations to check results.

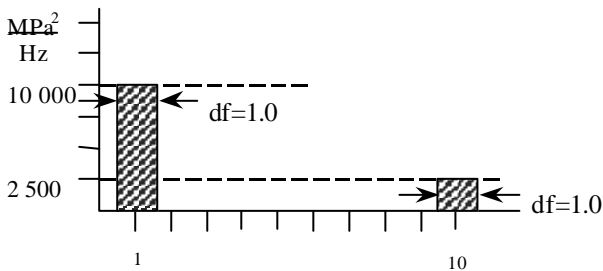


Figure 80. A simple 2 peaked PSD

### 8.5.1 Time Domain By Hand.

An approximate visualisation of the original time signal (represented by the PSD in Figure 80) can be obtained by adding together two sine waves, one for each block in the PSD, where the amplitude of each is obtained (approximately) from 1.41 times the root mean square (rms) value (see Figure 81). Since the rms of each block can be calculated (approximately) from the square root of its area we get the following stress ranges:

- ◆ Sine wave 1 at 1Hz with a stress range of  

$$\sqrt{10,000} * 1.41 * 2 = 282 \text{ MPa}$$
- ◆ Sine wave 2 at 10Hz with a stress range of  

$$\sqrt{2,500} * 1.41 * 2 = 141 \text{ MPa}$$

(N.B. Stress Range = 2 x Stress Amplitude)

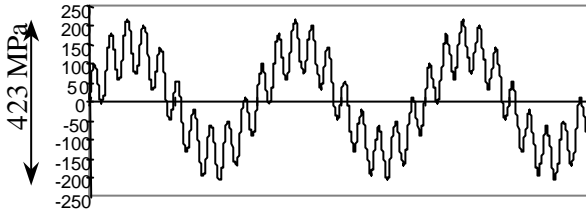


Figure 81. A time signal reconstructed from the 2 peaked PSD

If a Rainflow counting procedure is adopted the 10Hz cycles are extracted, unaltered at 141MPa. This then leaves the 282MPa cycle compounded with the 141MPa cycle to create a Rainflow cycle of range 423MPa. The Rainflow count is therefore,

Table 1. Rainflow cycles counted from the simple time history

Number of cycles per second	Total (Rainflow) Cycle Range
1	141MPa
10	423MPa

If we use a typical steel with S-N data of the form

$$N = 1.0E+15 * S^{-4.2} \text{ we get,}$$

Table 2. Allowable N obtained from the S-N curve

Rainflow Cycle Range (Si)	Cycles to Failure (Ni)
141MPa	9.4E+5
423MPa	9.3E+3

An approximate Palmgren-Miner damage calculation on the time signal then gives;

$$E[D] = \frac{10}{9.4E+5} + \frac{1}{9.3E+3} = 1.18E-4$$

This corresponds to a fatigue life of **8462 secs**

### 8.5.2 Frequency Domain By Hand.

Moments can be computed easily from the PSD using the expression given earlier.

$$\begin{aligned} m_0 &= 12\,500 & m_1 &= 35\,000 \\ m_2 &= 260\,000 & m_4 &= 25\,010\,000 \end{aligned}$$

From which we can compute

$$\begin{aligned} E[0] &= 4.6 \text{ upward zero crossings per second and} \\ E[P] &= 9.8 \text{ peaks per second} \end{aligned}$$

$$g = 0.465$$

$$\text{rms} = \sqrt{m_0} = 112 \text{ MPa}$$

From which we get an equivalent sine wave magnitude  
 $= 112 * 1.41 * 2 = 315 \text{ MPa}$ .

For this,

$$N(315\text{MPa}) = 3.2E+4$$

from which we can get a fatigue life of **3265 secs**

### 8.5.3 Computer Based Calculations Using MSC.Fatigue

Fatigue life using Narrow Band formula **1472 secs**

Fatigue life using Dirlik **7650 secs**

The correlation between the result obtained by hand (8462 seconds) and the Dirlik result (7650 seconds) is very good considering the simplifying assumptions that have been made. The hand calculation made with the narrow band assumption

(3265 seconds) is also acceptable when compared to the computer based Narrow Band result (1472 seconds). As we would expect, the narrow band solution is conservative.

## **8.6 An FEA Based Example**

There are many practical applications of the above techniques. Figure 82 shows the reliability of a solder joint in a Boeing Patriot Advanced Capability Missile Electronic component using MSC.Fatigue.

This was published at the MSC Aerospace User Conference, California, 1999. Frequency response and random vibration analysis was performed using MSC.Nastran to extract transfer functions due to 1G accelerations. The solder joints were modelled using 8 noded brick elements. Acceleration input load PSDs were defined based on measured vibration test and flight worthiness levels. Stress response PSDs were extracted to determine fatigue lives based on the Stress-Life approach. Figure 82(e) shows the fatigue life contour plot.

## **8.7 FE Hints and Tips – Vibration Fatigue**

### **8.7.1 Calculation of Frequency Response Function (Transfer Function)**

One of the most important parts of a vibration fatigue analysis is the calculation of transfer functions. With NASTRAN this is called a frequency response analysis and the process is shown in Figure 83. The use of transfer functions is described more fully in section 8.2.5. It is clear that the values of the output PSD are directly dependent on accurate values for the transfer function at points 1 to 6. So in this case the minimum number of points required in the frequency response analysis would be 6. Obviously as the analysis gets more complicated then more points are required, especially as the number of modes in the response increases and also where function values are curved rather than linear. Therefore, in order to improve accuracy, it is better to increase the number of frequency response points as much as possible. However, every point requires an additional solution in the frequency response run so a compromise has to be adopted between computational run time and numerical accuracy of the final result. Because of this possible loss of accuracy great care is needed with this step.

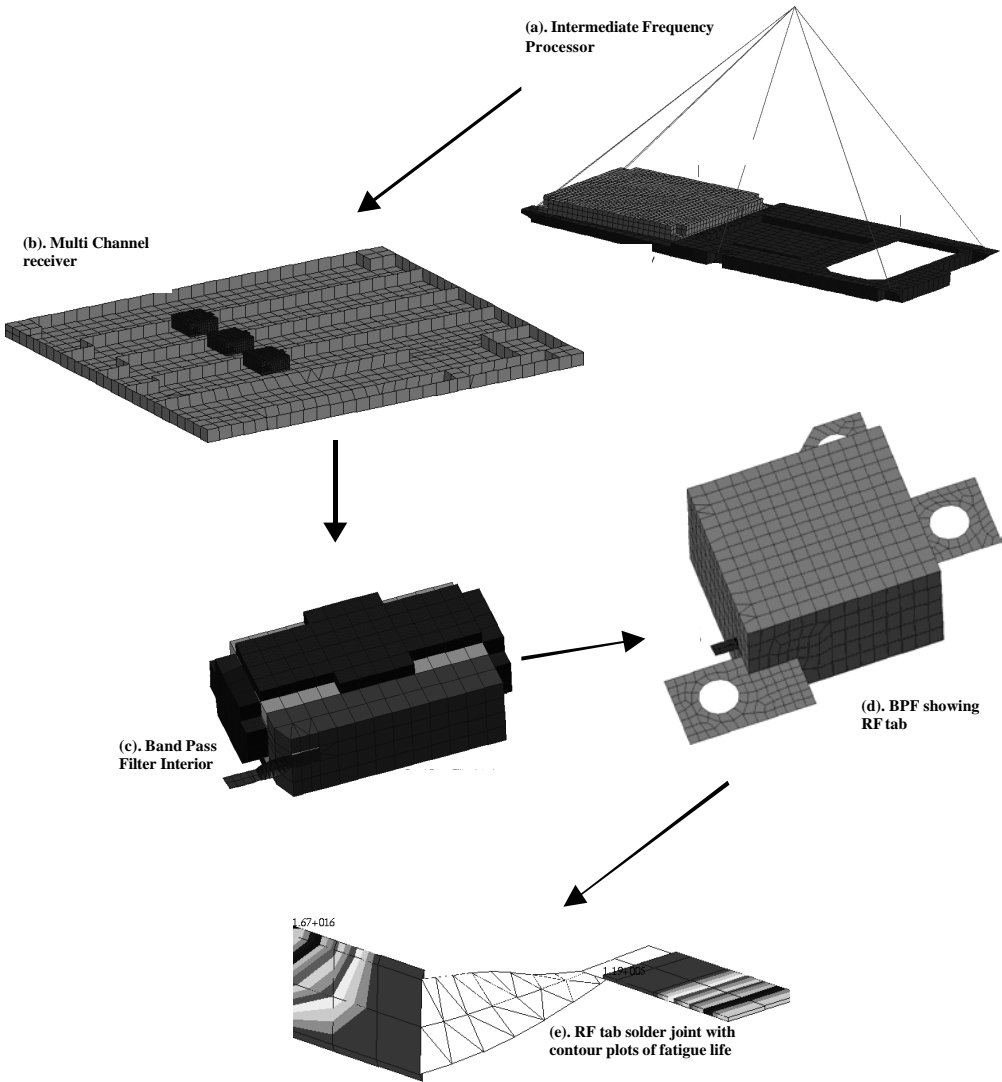


Figure 82. A solder joint in a Boeing Patriot Advanced Capability missile



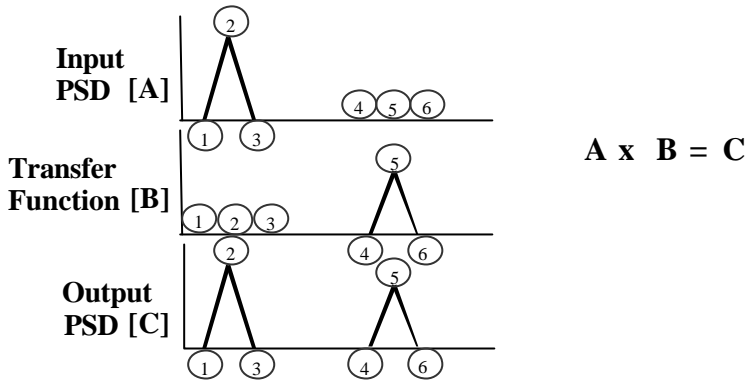


Figure 83. The process of calculating a transfer function

### 8.7.2 Verifying Vibration Fatigue Results

It is always advisable, irrespective of the analysis being undertaken, to check the computer-based result using, if possible, a hand calculation. With vibration fatigue calculations using PSDs this is also possible. One easy to follow procedure is to roughly estimate the PSD area and then take the square root of this. This gives the rms value (see section 8.1.2). There is a general rule of thumb that works quite well which is based on the assumption that the approximate maximum range of stress present in the signal is 6 times the rms value. This stress range value can then be compared against an appropriate material SN curve to obtain an approximate fatigue life. In order to obtain a value for the applied number of cycles the dominant frequency in the PSD can be used.

# 9 FE Model Building and Post Processing Issues.

## 9.1 Introduction

The incorporation of fatigue life calculations into the FEA world has brought significant advantages to the designer, not least of all the ability to do up-front fatigue calculations long before a prototype exists. However, the combination of the two technologies also poses challenges. Some eminent workers in the field have proposed the advisory philosophy that the FE step should be considered as the pre-processing stage to the fatigue life prediction and not the converse, ie that a fatigue life prediction is the post processing stage to the FE analysis. This helps to focus the need to ensure careful FE modelling etc. The aim of this section is to highlight some of the more important aspects that can be encountered.

## 9.2 Process Issues

The following items are intended as a checklist on the whole process.

***Step 0. Identify design and analysis criteria up-front.*** The first step prior to building a model is to understand the objectives of the analysis, how the component is used in service, how it is loaded, how it interacts with the environment. It is better to go in with a predefined approach, then challenge it as the risk, cost and weight implications of a deficiency are assessed. Too many analysts only see the maths behind the model, not what they are trying to represent.

***Step 1. Collect/Determine material parameters.*** Design parameters should always have an identified allowance for conservatism; safety factors should be noted as either statistically justified for known material variability or as experience based.

***Step 2. Build FE model and check global FE modelling/meshing quality.*** This must include the manner in which the non-structural masses are represented: point mass, mass moment of inertias, attachment points to model, etc. Convergence should be checked and error checks performed for stress-strain jumps between elements in areas of concern to ensure good local convergence. Fatigue assessment

accuracy is dependent upon good local convergence as well as adequate global convergence. A sub model may be necessary to give additional accuracy.

***Step 3. Identify all important free body loads acting on the system/component.***

Consider whether the structure can be analysed statically or whether dynamic effects are significant. If dynamic, is it a very large FE-model? In which case super elements, modal transient response calculations, or random vibration analysis may be applicable. If it is not a large model, direct or modal transient response calculations may be viable. Of course, the definition of a large model will vary.

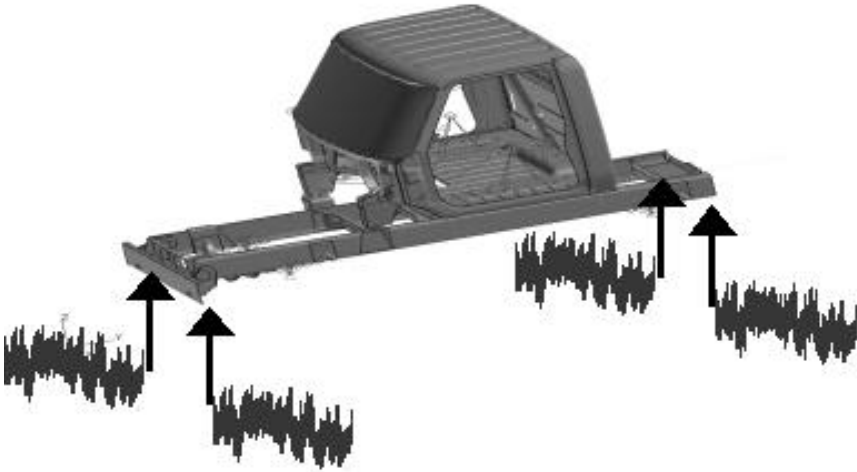
***Step 4. Partition these loads into an applied set and reactions set and determine type of analysis.***

The main issues of importance here concern whether the model is constrained or not, whether the response might contain some dynamic elements and how many load inputs are present. For static models constrained by the boundary conditions most of the work can be left up to the FE solver. The solver will determine reaction forces caused by the applied loads set and ensure that these are in balance with the applied set. Some checks for this are advisable. Restraints points must be realistic and not over restrained, e.g. Poisson effects. Where static models are unrestrained a concept called ***Inertial Relief*** is often adopted. The concept uses the inertia (mass) of the structural elements as the reaction forces to keep the structure in equilibrium. Again the basic rule is that equilibrium must be maintained. Where an unconstrained dynamic model exists there are 2 main methods that can be adopted, these being the ***Large Mass*** method and the ***Lagrange Multiplier*** method. The large mass method is the most common approach for simulating an acceleration input. With this approach a force is applied to a very large mass placed at the loading points. The force is varied in order to recreate the desired acceleration input time history at the loading point. Of course, in this case, equilibrium is maintained through the counter balance of the inertias from the real mass present in the model.

***Step 4a. Multiple inputs for dynamic models.***

Consider the case where there is more than one input loading to an unconstrained model such as the simulation of multi input acceleration loading to a road vehicle on a four wheel test rig (see Figure 84). In this case there is always some uncertainty about boundary conditions because the load application points are also, to some degree, the constraint positions. Typically, for a dynamic analysis, the characteristics of the model for each acceleration input will be determined one at a time by applying appropriate unit inputs to one load application point whilst applying appropriate boundary conditions to the other three. In general, the recommended practical approach is to refer back to the actual test rig situation. In this case, with one load cell applying an acceleration with the other three load cells turned off. These are the boundary conditions that should be recreated in the FE model. In other words, pinned joints with some degree of stiffness to simulate the true response behaviour of the actuators.

**Step 5. Perform FE unit load case analyses for each of the applied load set components.** Some care is needed here. It is reasonable to assume that, for a linear analysis, unit load cases will give the same relative result as those given by more realistic load cases. After all, it is the response per unit input that is important. However, with some commercial codes there is a numerical round off problem that can be encountered when you combine (multiply or divide) two numbers of very different magnitudes. For this reason some commercial codes recommend using realistic values for the “unit” load cases.



*Figure 84. Four point acceleration to a vehicle.*

**Step 6. Apply external loads and cross reference to FE (unit load case) results for all input time steps.** At this point each FE result has to be tagged with the same reference that applies to the relevant external load case. This is the step where, for a static analysis, the actual loading time histories applied to the model are combined with the stress values obtained from the unit load cases referred to above. For this static case, linear superposition is used to combine the effects of more than one loading time history input.

**Step 7. Extract stress time histories for all time steps.** This can either be done for various critical regions, or over the whole model.

**Step 8. Perform fatigue analysis.** This will be done with the chosen fatigue analysis approach. The analyst (engineer) must be very cognisant of each of the analysis settings he is employing. Frequently, default settings are used without question. Remember that most engineers are jacks-of-all-trades. It is therefore very important to question everything that is done; even rules of thumb do not always apply.

**Step 9. Plot results.** Guideline for results visualisation are given in section 9.9.

**Step 10. Identify critical locations.** Notice that fatigue ‘hotspots’ may only coincide for simple loading and geometric cases. Shortest lives could occur at other locations, e.g. due to combined loading effects, or surface finish conditions.

**Step 11 Reassess local FE modelling/meshing quality in critical locations and optimise if necessary.** Consider bi-axial stress conditions and vector mobility. Use multi-axial fatigue techniques if necessary. Reconfirm any relevant modelling assumptions.

**Step 12. Perform local fatigue analyses using stress hot spots (local regions of high stress). Perform sensitivity studies and assess quality of result.** Sensitivity analyses can (and should) be conducted for many of the inputs in the life calculation. For instance, how much would the stress have to vary to alter my conclusion? Is my mesh good enough? Are my boundary conditions reasonably correct? Are my loads reasonably correct? Is my life goal reasonably correct? Are my material properties, and other fatigue analysis assumptions reasonably correct? What is the worst case, or best case, that could result from possible ranges in these parameters? Does the issue of variation in predicted life become a moot one if a simple, cost effective design change can be implemented? What was the predicted life relative to the established loads-goals criteria? Would a reasonably small change in stress change the conclusion? Was the peak stress beyond yielding? By how much? What were the stress-strain/life predictions for the previous design or current product at the comparable location?

**Step 13. Consider past experience.** Correlation of analysis of similar designs is a crucial step. This helps to identify appropriate techniques and robustness of the analysis. Did the previous design or current product crack in lab or field tests? Has the geometry changed? Is the location sensitive to more than one load case for this condition, or sensitive to more than one condition? For instance, a load condition might be vertical loading, and a load case within a condition might be where the load was applied or how it was distributed. Did the modelling assumptions influence results? What is the sensitivity of results relative to location of loads and restraints? Are any aspects not modelled properly, etc? In other words, a location that was over stressed for just one conservative load case, might be concluded to be marginal or even acceptable overall if there has been no past experience of problems in similar designs. Or, alternatively, a location that (a) was similar to a crack location in the previous design or current product, and (b) was predicted to be acceptable, but (c) would be marginal for minor adjustments in loads, might be concluded to be marginal.

**Step 14. Determine whether or not objectives have been met!**

## 9.3 Model Building

An FE analysis may be carried out for a number of different purposes, and the modelling requirements depend on its intended use. For fatigue analysis, the results are very sensitive to the accuracy of the calculated stresses and strains in localised regions of a component. To achieve acceptable levels of accuracy, the following are essential requirements.

### 9.3.1 Requirements

- (a).* The component must be represented accurately, both in terms of its physical behaviour and the material characteristics. Where smaller models are required, higher order elements should be used if necessary to more accurately follow the physical contours of the component.
- (b).* Externally applied loads, and constraints, must also be represented accurately. Apparently insignificant changes to the way the loads and constraints are applied to the FE model can make surprisingly large changes to the deformation and hence the strains (see section 9.10).
- (c).* Shell elements must be used with care, and in particular, only where the structure is one which can reasonably be treated as a shell (i.e., where the thickness is small compared to significant geometric features). Also, depending on the assumptions of the elements, some FE solvers do not calculate, or return, out-of-plane strains for shell elements making the calculation of strain invariants (von Mises, principals, etc.) impossible. Therefore only stresses should be used from shell elements.
- (d).* It is important that elements are chosen with a view to generating accurate grid point stresses and strains as fatigue cracking usually starts at free surfaces and edges. In general, better results are likely to be achieved by using higher order elements, even if they are fewer in number. Use of higher order elements also permits better representation of geometric features.
- (e).* Ideally, the mesh should be refined to a point where further refinement produces little change. The criterion used must be local stress and strain and not global stiffness. There is little to be gained by excessive refinement in non-critical areas. The sole requirement in these parts is that they transfer loads correctly to the critical areas.
- (f).* Use of triangular and wedge elements should be minimised and care should be taken with aspect ratios. The effects of joins between elements of different types and shells of different thickness need to be carefully considered as these have the capacity to act as fictitious stress raisers.

(g). Wherever possible, verification of the FE calculated strains should be made by comparing with strain gauge measurements, or alternatively, predicted static deflections should be compared with measured ones.

(h). Gap elements in models must be dealt with by treating them as if they were transient analyses. It is really a load step analysis. You can't use the pseudo-static method here at all. A non-linear load step analysis simulates the contact using the gap elements. So all the external loads have to be applied simultaneously to the same analysis and a stress history at the interested locations must be supplied to the fatigue analyser.

### 9.3.2 Observations

In reality, localised plastic zones and radii at discontinuities in the surface mean that excessively high SCFs are less common than some FE models would suggest. For example, consider the situation shown in Figure 85.

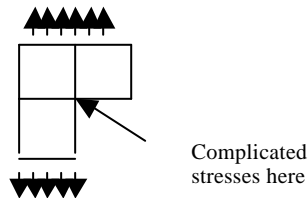


Figure 85. Artificially high stress concentration factors

Another related situation concerns facetting along a 3D edge. How do we deal with this in FE modelling for a fatigue analysis? The answer is that the designer has to use some engineering judgement in looking at his results. He should ask the questions what features should be removed, and which features should be retained?

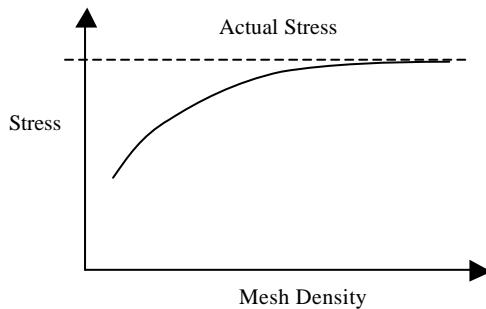
Regarding defeaturing. It may be helpful to think of stress as a flow problem. Think of flow from a small pipe to large pipe. The fluid can't immediately turn the corner at the diameter change, and only fills the pipe downstream (or something like that). In a stress problem, the analogy would be that some of the material in the big pipe near the diameter change could be defeatured because the stress can't flow into that area. Stress concentration charts could be useful. They can provide information about how approximate a filleted surface can be made, or some other change, before an unacceptable change in the stress concentration factor occurs.

Think which features need to be retained. These might include internal details, such as fillets, notches and small holes. If this level of detail is impractical to include on a large FE model, then they should be simplified and then included as

discrete corrections in the fatigue analysis. A good fatigue analysis system would allow the engineer to 'zoom in' on a node and apply additional stress concentration factors locally.

### 9.3.3 H vs P-type element choice

Stress accuracy for conventional H-type finite element analysis is highly dependent on mesh size – with a finer mesh improving accuracy (see figure below). The choice of mesh size is down to the experience of the user, and unless several iterations of the same model are analysed, an appropriate mesh density is difficult to define. The available computer resources often limit the degree of mesh refinement, and only rarely can the analyst pursue a quantifiable convergence.



*Figure 86. The concept of stress convergence against mesh density*

One possibility is to use geometric P-type methods. It has been suggested that these have two benefits over H-type. The first is the way that the elements can more easily follow the underlying geometry, as opposed to H-type elements which 'facet' curved surfaces. The second is that the stress accuracy is not dependent on the density of the mesh, with the polynomial order of each element being progressively increased with the P-type solver iteration – converging to a pre-defined error tolerance. However, element shape does have some effect on solution path and so element shape and the mesh is still important. It is also important to be careful with point loads and other singularities. This can require a strong understanding about the P-type element method

As fatigue life is highly sensitive to stress levels, stress accuracy is more critical for fatigue applications than for any other analysis type. Therefore, for some types of models P-type analysis methods might have advantages over conventional methods. However one practical issue with P-type elements is that the polynomial order is arbitrary, so the converged order may vary in all directions. Standard post processors such as MSC.Patran don't have the ability to handle arbitrary polynomial orders so often the stress field has to be discretised into a standard H type grid for post-processing.



## 9.4 Meshing

It is not possible to make general statements about which elements to use. Some would say that 4-noded tetrahedrons are no good. However, the important point is that they are constant stress and sensitive to shape. A billion well shaped 4-noded tetrahedrons should provide very good results. But in general, since fatigue analysis requires a high quality stress analysis, elements such as quadratics probably become preferable. Beams are not really suitable, as they will not model the details that cause fatigue, such as the mismatch at a joint. Shells are useful as a modelling technique, but require more care when post-processing. Generally, one should provide sufficiently refined meshing around areas of real stress concentration and material discontinuity, in order to recover accurate surface stresses.

In stress concentration areas the mesh quality should be high; this implies a preference for HEX elements with quadrilateral shape functions and good shape characteristics. Away from these areas TET/WEDGE/coarse meshing is more generally acceptable. The use of a coarse mesh for preliminary analysis to identify critical locations is beneficial. Higher order TETS are generally regarded as satisfactory.

## 9.5 Dealing with Loads.

New products are often numerically analysed long before the first prototypes are built. This means that "real" load data does not exist. Calculations can be made using similar signals from a previous product generation, sometimes after some scaling. Load signals are, however, quite often calculated using commercial multi-body codes such as ADAMS and DADS. Input can then be, for instance, road topology. This is an extremely important topic. Calculated results will never be better than the input data.

Regarding point loads, these represent a mathematical singularity, and the results near to the load application point cannot be trusted. If in doubt, more accurate load distributions near the location of application should be used.

Somewhat similar comments could be made about beam attachment points. However, if the nature of the structure suggests that the area of attachment could be highly stressed, a more refined model, or even just a hand calculation given the output beam loads, should be used.

With regard to loading sources some standards do exist, but improvements in engineering design move faster than standards so it is likely that engineers will want to use more specific data as input to their fatigue analysis. Standards, such as

ISO 7096 for earthmoving seating, can of course be used where no other data exists or where legal requirements dictate. However complying with a standard does not guarantee trouble free service operation, or remove professional responsibility.

One possibility often suggested is to use one of the many published random loading sequences such as CARLOS, TWIST, or one of the SAE sequences like SAETRANS. If this approach is used care must be taken to identify objectives. These sequences have a history of intensive development dating back to the 1960's. Their main purpose is to improve the accuracy and reliability of testing, or simply as a representative comparison. If Miner's Hypothesis is accurate and reliable any short sample of one of these sequences could be reduced to an equivalent number of cycles at any chosen amplitude, which could then be used in testing or in an FEA calculation. The sequences are used in testing because we know that Miner's Hypothesis is not totally accurate. Two of the sources of this unreliability are failure to allow for interaction between the different stress levels present and failure to allow for sequence effects (see section 4.6). Using generic loading sequences with FEA is only worth considering if the fatigue model being used allows for either or both of these. The most common case is a calculation using a sequence-dependent Crack-Propagation model like the Willenborg one, but some Strain-Life calculations will also qualify.

As stated above, there is some movement towards using analytically based load sequences, but it is too early to assess the contribution such sequences will make. The usefulness will differ between sequences. The SAE ones were derived to ensure consistency in a collaborative testing programme, and are too short to be useful. Longer ones like the aerospace TWIST and FELIX, the offshore WASH and automotive CARLOS are still intended for testing, the objective being to rank materials and fabrication methods in order of merit. They may still prove useful in FEA, though, because they are available before prototypes have been made. They can aid the engineer by allowing analysis of sensitivity to such things as changes in material properties, and they may prove useful if testing is being used to refine a model. In fact, fatigue analysis and FEA can even help here by evaluating various proposed loading sequences. If different methods, which may not give the same result, still lead to the same engineering conclusions then the most appropriate one can be used with confidence, at least in initial studies.

## **9.6 FEA Based Global Analysis Options**

### **9.6.1 Analysis Options**

There are a number of FEA based methods for obtaining the stress information that is required to perform a fatigue life calculation

(a). **Static** structural (and fatigue) analysis can be undertaken utilising superposition capabilities for combining multiple load application inputs. Unit inputs of load are applied to all desired load application points. The resultant stresses (caused by the unit load cases) are then factored by the actual time history of loading for that load application point. This process is repeated for all load application points and the results are linearly superimposed. Fatigue life calculations are then performed using these combined stress histories. This method ignores dynamic influences such as mass effects.

(b). **Dynamic Transient** analysis. If this approach is used, the stress histories are produced at each point of interest using an FEA transient analysis method. These stress histories are also superimposed to obtain the required combined stress histories, but the FEA solver handles this. Fatigue life calculations are then performed on these resultant stress time histories. This method accounts for all dynamic effects but is less versatile in that all loads must be combined in a single FE analysis. The same approach can also be used for ‘static transients’ such as a gap closing problem, and thermally driven transients, assuming that appropriate materials data is available

(c). **Frequency Response** analysis. In this approach the transfer functions are produced using the desired solver. These transfer functions are then resolved onto the desired stress axis system (usually principal stress). The response caused by multiple random loading inputs is then obtained using standard random process techniques. The effect of correlation between inputs can be dealt with by including Cross Power Spectral Density functions in the input loading data. This method accounts for all dynamic effects and is quite versatile.

(d). **Random Vibration** analysis. In this approach the response Power Spectral Density function is determined directly from the FEA solver. Effects due to multiple load inputs must be dealt with in the FE analysis as with a transient analysis approach. All dynamic effects are accounted for but this method has the limitation that fatigue life can only be computed for a single component direction. Stress response results are not resolved onto a desired stress axis system by the FE analysis.

(e). **Modal Transient/Superposition**. A hybrid method called modal superposition can be adopted, with or without mode removal, as described in section 8.3. The advantage of modal superposition when compared to direct transient response is that less data need be retained in order to reproduce the time signals. This is of course dependent on the number of modes retained.

## 9.6.2 FE Results output type

It is appropriate to review the manner in which results are stored in an FEA database in order to avoid confusion as to what results type will be used in a fatigue analysis.

Results stored in the database can be associated with either the nodes or the elements of a model. When the results are associated with nodes each node will have six (6) component stresses or strains (assuming 3D) when considering tensor results.

Results associated with elements have element positions defined. This means that there will be multiple results for each element. These element positions can be the nodes, Gauss points or the element centroid, depending on the FE system and output options specified in the FE analysis. If only one element position exists, then generally this means the results exist at the element centroid in which case each element will have six (6) component stresses or strains. If more than one element position exists then there will be six (6) component stresses or strains for each element position for each element.

In addition, multiple layered results can exist for results associated with both nodes and elements. When multiple layers exist, it is necessary to select the Surface as either Top or Bottom. Top is often the default if no Surface option is selected. Multiple layers are often used in composites.

## 9.7 FE Based Local Analysis Options

### 9.7.1 What is “Absolute Maximum Principal” Stress and “Signed Von Mises”?

Cracks (other than very small cracks) tend to be created and driven by a dominant stress acting perpendicular to the crack direction. Consequently, an absolute maximum principal stress parameter is likely to give a sensible indication of the crack driving force. A signed maximum principal (or absolute maximum principal) stress is also a sensible parameter since cracks grow in proportion to the stress range, i.e. from an absolute minimum to the maximum stress. In practice, the engineer should assess the life on at least two stress combination parameters. The most appropriate parameter may be gauged from the local stress directions.

The following example explains what is meant by "Absolute Maximum Principal" stress. For example, if the time series were:

Time	0	1	2	3	4	Max range
Max Principal	100	-100	200	-200	500	-200 to 500 = 700
Min Principal	50	-150	-500	-250	-10	-500 to 50 = 550
Abs. Max. Principal	100	-150	-500	-250	500	-500 to 500 = 1000

You can see that using the Absolute Maximum Principal gives a larger outside stress range than just using Maximum or Minimum Principal. Remember that "Absolute" does not mean that we take the sign away. We determine what the largest absolute value is and then take that number as it is, leaving the sign.

Also, when using other stress combinations such as von Mises or Shear/Tresca, these values tend to be always positive which halves the actual stress range. Therefore the sign of the Absolute Maximum Principal should be utilised to "sign" the von Mises or Shear/Tresca values. So, if the Absolute Maximum Principal is negative at that point in time, so is the calculated von Mises value. Von Mises and Tresca are popular for stress analyses, but not recommended for fatigue analyses. It is non-directional, whereas fatigue cracks are directional.

### 9.7.2 The Choice of Element Centroid Versus Nodal Results

Fatigue cracks normally initiate and grow from free surfaces. It is therefore vital that, whatever system is used, the stresses at the surface are obtained.

Element centroids are somewhat easier to deal with, especially for shell elements because there is one stress value per element which can be used for fatigue analysis. For nodal results, some averaging has to be done because, in general, each node will be associated with multiple elements and this adds complexity. Results need to be averaged in a consistent co-ordinate system and by default shell stresses are in a local co-ordinate system for each element. Also this averaging may reduce or 'wash-out' maximum stresses coming from one element and affect the predicted fatigue lives. However, others say that the averaging does a nice job of covering up the sin of using poorly shaped elements in real world models. FE codes like MSC.Nastran have grid point stress options that perform this averaging but situations such as where two shells meet at 90 degrees can still cause concerns.

For solid elements, element centroids are not generally acceptable because fatigue problems and maximum stresses tend to occur on the surface. An alternative to nodal averaging is to skin the solid with very thin shell elements and recover the element stresses in these shells. The shells should give the surface stresses, which can then be used in a fatigue analysis.

## 9.8 Pre Processing of Loading Data

Engineering loading information will consist of two main types, time signals and PSDs. With both it is important to ensure that frequency information is correctly conditioned before use. For instance, sometimes high frequency signals need to be filtered to remove noise prior to use. This same requirement can exist for time signals. Figure 87 shows how spurious spikes can be removed in the time domain

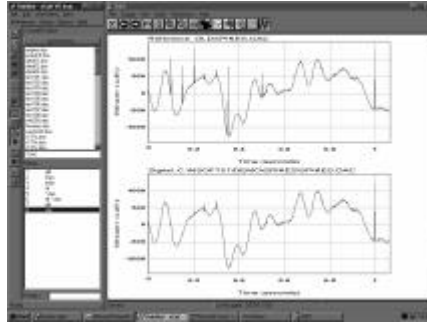


Figure 87. Cleaning up acquired time history data

## 9.9 Post Processing of FE Based Fatigue Results.

Output data from a fatigue life analysis falls into 2 categories.

**Global** multi location data can be presented through the use of contour plots. Examples of what can be plotted in this form are, (i) fatigue life, (ii) fatigue damage and (iii) bi-axiality (a measure of the multi-axial stress state). An example of a fatigue life contour plot is given in Figure 88.

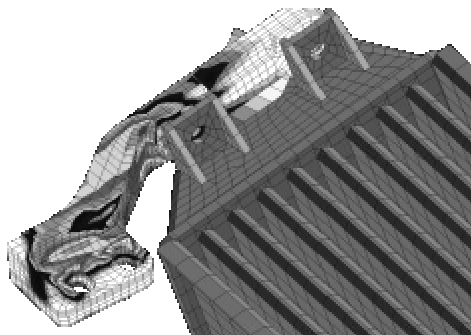


Figure 88. A Fatigue life contour plot

Local *single location* fatigue analysis data typically comes in the form of a histogram of Rainflow range cycles or damage as shown in Figure 89 and Figure 90. Figure 91 shows the combined plot.

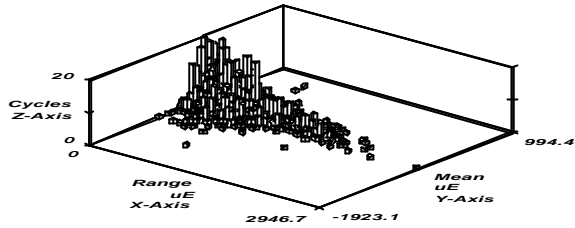


Figure 89. Rainflow cycle histograms

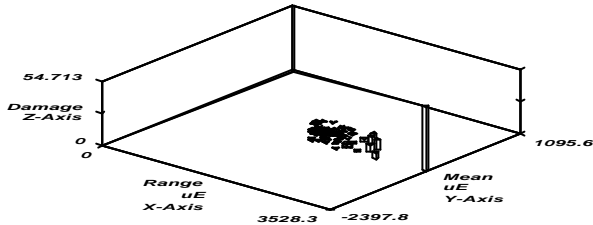


Figure 90 . Damage distribution for each Rainflow cycle

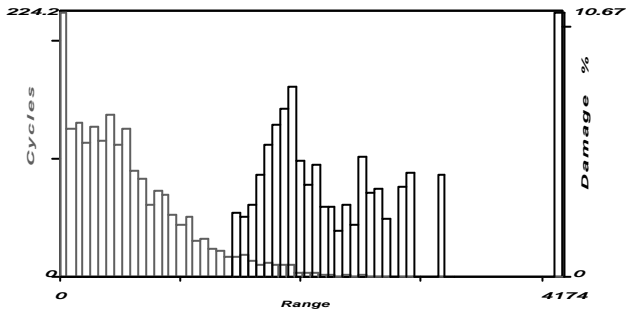


Figure 91. Joint plot of damage and Rainflow functions

One should be careful about using continuous tone contour plotting, it tends to mask stress gradients near areas of stress concentration. Vector stress plots may be useful in understanding stress flow. Contour plotting of fatigue lives can be misleading, especially for the uninformed. If contours are necessary (assess the rate of change, how large a zone is below the design life, etc) be careful about using a

uniform range, or logarithmic scale. It might be better to use a linear scale with upper bounds, so that very long lives are not contoured at all.

Always look at the principal stress vector directions. The vectors should confirm any cracks found and the direction in which they propagate. This is simply a way for engineers to compare test and analysis. Plot both Pmax and Pmin vectors together. This gives an idea of the bi-axiality conditions. Note that under fatigue loading these will vary with time. The FE loadcases only represent a snapshot. The engineer needs to consider these issues too.

It is always a good idea to check that the third principal stress is (near) zero at free surfaces. Simple mesh studies can be useful. Plotting un-averaged nodal stresses is a good way to check results quality.

## **9.10 Accuracy of Fatigue Life Estimates**

Traditionally, the Stress-Life approach to fatigue life estimation is associated with a probability of survival based on the statistical significance of the Stress-Life curve drawn through the experimental Stress-Life data. The same approach can be used for the Strain-Life method. However, the experimental materials data from which the Strain-Life curve is obtained exhibits substantially less scatter than is experienced with Stress-Life data. This is primarily due to the fact that Strain-Life data is obtained from a test where the control parameter is the strain in the region where the crack is being created, i.e., the test is fully constrained. Strain life based fatigue life estimates are typically accurate with a factor of two (i.e. one half to two times the actual life).

Fatigue life predictions are, like any other simulation, dependent upon the accuracy of the data put into the process. However fatigue calculations tend to exaggerate inaccuracies because of the logarithmic nature of fatigue; if my stresses are wrong by a factor of 2, the fatigue life predicted could be wrong by a factor of 1000 or more. This means that care does need to be taken with the quality of the stress and load information used.

There is by nature a lot of scatter in fatigue and if nominally identical components are physically subjected to the same loading environment, lives may be different by a factor of 2 or 3. Defining the required loading environment for a system such as a vehicle is not trivial and is further complicated by issues of statistical variability and repeatability. However, where the system and environment can be well defined, good correlation can be achieved between predicted and measured fatigue lives. Experience on vehicle programs has shown correlation in fatigue lives within a factor of 2-3 between test based fatigue lives using drive signal measurements



from a proving ground and FE calculations using standard commercial codes, which is probably within the repeatability of the test.

The methods for improving fatigue life are dependent upon the stage in the overall process. Early in the process, potential fatigue problems can be avoided by geometry changes such as increased radii. Later in the process, such changes may be prohibited by space or tooling implications and other changes may be considered such as surface treatments like shot peening or a different grade of material. Increasing the thickness of a component is an obvious way to reduce stresses but this can be inefficient in weight terms. Adding reinforcements may be more efficient but often adds production cost and complexity and can also result in chasing the fatigue problem to the next weak point. Another way of improving fatigue lives is to reduce the loads and in the case of a vehicle this can be achieved by changing system parameters such as suspension and bushing stiffness to reduce the transmitted forces.

## 9.11 General Conclusions

This chapter has listed points that must be considered when using FEA as a tool to help fatigue life estimation. Inevitably it has highlighted the difficulties that are likely to arise. Against this, though, we need to set the benefits. It has been stated many times in this manual that fatigue starts at local places in a component, and stress or strain histories at those local points are the key to accurate life prediction. Traditional design has usually achieved this by calculating nominal stresses at some point convenient to the analysis, and then applying some multiplier to get the local history. The dominant source of these multipliers has been tables of stress concentration factors,  $K_T$ . Because FEA is capable of providing local stresses this use of  $K_T$  can be eliminated. This inevitably throws more responsibility on the analyst, which is one of the reasons why more care must be taken to cover the points made in this chapter. The reward, though, is that geometries, which can only be handled with difficulty using traditional methods, can now be treated as routine cases. Another reason why increased caution needs to be exercised when extending FEA from static to fatigue design is the dependence of life on high powers of stress. This means that differences that would be acceptable in static analysis now cause unacceptable errors in life prediction. Finally, the importance of dynamic behaviour when fatigue is being considered means that input loads and reactions must be rigorously documented and classified, including, in some cases, their frequency content.

# 10 Bibliography

No attempt has been made in the text to give complete references to the extensive literature that exists in the fields of fatigue of materials and FE analysis. Instead this chapter will give a short list of useful sources that are often available in technical libraries. When a particular technique is described in the text, for instance predicting Crack-Propagation using fracture mechanics, the name of an author suitable for use in a literature search will usually have been mentioned, in this case Paris or Erdogan. This can then be used in a literature search if required.

## **General References – FEA**

The best source of practical information concerning FE analysis is either one of the many NAFEMS publications on the topic, or one of the proprietary FE analysis software vendor product reference texts, such as the MSC.Nastran or MSC.Patran user manuals.

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